Four Essays on Language Competition and Dynamic Language Policy Evaluation

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Abstract

This thesis deals with language policy evaluation from a language dynamics modeling perspective. As linguistic diversity is an essential feature of most modern societies, states and administrations have to thoroughly design and analyze language policies. Potential effects, benefits and costs have to be assessed and weighted against one another. A pivotal characteristic of language policies is that the numbers of their beneficiaries and costs can change dramatically over time. To account for these changes, the thesis proposes a combination of traditional policy evaluation techniques with well designed language dynamics models. In contrast to previous models in the literature, the thesis proposes and analyzes models based on parameters obtainable from empirical data. It is argued that this is a prerequisite to analyze the long term effects of policies in a realistic fashion. This thesis consists of four self-contained essays. In the first essay we show with the help of an abstract model that it can be optimal for the state to keep a minority language alive in the form of bilingualism. In the next two essays more realistic models are developed and applied to the empirical cases. In the last essay extensions of the previous models to the case of several minority languages are presented.

Zusammenfassung

Diese Dissertation beschäftigt sich mit der Evaluation von Sprachpolitiken mit Hilfe von Sprachdynamik Modellen. Da sprachliche Diversität ein zentrales Merkmal moderner Gesellschaften darstellt, müssen Staaten und Administrationen Sprachpolitiken sorgfältig gestalten und evaluieren. Mögliche Effekte, Nutzen und Kosten von Politiken müssen bewertet und gegeneinander abgewogen werden. Eine wichtige Eigenschaft von Sprachpolitiken ist dabei, dass sich die Anzahl derer, die von ihnen profitieren, sowie deren Kosten über die Zeit stark verändern können. Um dies zu berücksichtigen, präsentiert die Dissertation eine Kombination aus klassischen Politikanalyse Werkzeugen und neuen Sprachdynamik Modellen. Im Gegensatz zu bereits existierenden Modellen, können in den neu entwickelten Modellen Parameter aus empirischen Daten geschätzt werden. Dies ist eine Voraussetzung, um langfristige Effekte von Politiken realistisch abbilden zu können. Die Dissertation besteht aus vier eigenständigen Aufsätzen. Im ersten Aufsatz wird mit einem abstrakten Modell gezeigt, dass es für einen Staat optimal sein kann die Minderheitensprache in Form von Bilingualität am Leben zu erhalten. In den folgenden beiden Aufsätzen werden realistischere Modelle entwickelt und damit zwei empirische Fälle analysiert. Im letzten Aufsatz werden Erweiterungen der vorherigen Modelle auf den Fall multipler Minderheitensprachen vorgestellt.

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1 General introduction

Linguistic diversity is an essential feature of most modern societies. This is apparent in countries like India, South Africa, Canada, Singapore or Switzerland. But even in seemingly monolingual countries, e.g. in Western Europe, there is a significant number of people who do not solely speak the official or majority language of the state but also have skills in one or several other languages. Such languages can be international vehicular languages (like English), languages historically rooted in (part of) the territory of the state (like Welsh in the UK, Basque in Spain, Hungarian in Romania), or the language of migrants and their descendants (like Turkish in Germany or Arabic in France).

This linguistic diversity has to be addressed and managed by the state. The state can not be fully neutral with respect to language. When publishing legal texts, designing education policies, offering public services or putting up street signs, linguistic decisions have to be made. That means that there is no pure *laissez-faire* option for dealing with linguistic diversity. "The correct opposition is therefore not one between linguistic freedom and linguistic regulation but rather between different forms of linguistic regulation. In other words, there is no zero-option in the field of language policy. We cannot not intervene" (De Schutter 2007, p. 17). Hence, language policies are unavoidable. But can they also be desirable? The question of desirability can be approached from an economic point of view. As illustrated inWickström *et al.* (2018), several language-related goods and services have the typical properties of collective or public goods and come with different externalities. The most important one is probably the network externality of language learning. Consider a person i learning language L. Part of the benefit she gets from learning L is that she can communicate with other speakers of L in this language. She decides to learn L if her individual benefits exceed her individual learning costs. If she decides to learn L, all other speakers of L can communicate with her in L as well. That is, her decision to learn L increases the benefits of all other speakers of L. But these benefits to other speakers of L are not taken into account in her individual decision on whether to learn L. "[T]he individual calculus here differs from the social one" (Wickström *et al.* 2018, p. 25). Consequently, spontaneous interaction can lead to sub-optimal results from a welfare point of view. To see that, consider the case where the learning costs of person i exceed her individual benefits ($c_i > b_i$). At the same time, the aggregated benefits from person i learning L to other people j in the population who speak L, $\sum b_j$, could be larger than the difference between person i's costs and benefits, i.e. $\sum b_j > c_i - b_i$. In this case, the other speakers of L could compensate person i. Without compensation, person i does not learn L (i's learning costs exceed i's benefits), but with the compensation she does (compensation plus i's benefits exceed i's learning costs). If person i is compensated and learns language L, then no one is worse off and some are better off $(b_i + \sum b_j > c_i)$. This illustrates that there are efficiency reasons for a state to engage in language policies. Another motivation for language policies can be fairness and equality between language groups.[1]

If the correct opposition is the one between different forms of linguistic regulation, then which linguistic regulation or language policy should be selected? As with any public policy, language policies come with potential advantages and drawbacks. To rank several policy options or to decide whether to implement a policy, these advantages and drawbacks have to be compared. "Public economics provides relevant frameworks to guide such choices", and the "analytical tool that suggests itself is cost-benefit (or cost-effectiveness) analysis" (Wickström *et al.* 2018, p. 10). Standard cost-benefit analysis monetizes costs as well as benefits and helps to compare both. If aggregated benefits of a policy exceed its overall costs, then the policy yields a potential Pareto improvement. Those who gain from the policy could – in principle – compensate those who lose from it, so that no one loses and in the end some are better off.

People can benefit from language policies in different ways due to the different functions of languages. Languages serve two basic functions: a communication function and an identity function. First, languages enable people to communicate with other people. They can read historical documents like books and diaries, they can talk to their peers or write pamphlets for future generations. The greater the amount of people they can communicate with in one language and the more domains of a society this language can be used in, the wider the communicative range of that language. Language policies alter both aspects, the number of speakers and the domains a language can be used in. If a language is taught in public schools, then this is likely to have a positive impact on the (future) number of speakers of that language. Hence, speakers of a language L normally benefit from L being taught at school. Language policies that enable individuals to use L in communication with the state (e.g. by providing forms in L), in court rooms or in hospitals, or that offer public services in L also enlarge the communicative range of L. This would allow speakers of L to use L in those domains, which is especially relevant if L is their native, first and/or preferred language. Hence, speakers of language L normally benefit from language policies that increase the communicative range of L.

Second, languages are essential carriers of cultural and ethnic identity. They often serve as markers of belonging to ethnic and national groups, are the medium of socialization of children, and give access to literature, music, history and discourse of ethnic and national groups.[2] Therefore, speakers of a language can also value language policies for identity reasons. Many policies that increase the communicative range of a language also increase its status in the society. This can be valued even by individuals who do not depend on the larger communicative range to function in society. Take a society with a majority language H that can be used in all public domains and a minority language L that is only used in private communication between speakers of L as an example. Assume that all native speakers

[1]See, e.g., Part II of Wickström *et al.* (2018).

[2]For a multidisciplinary discussion on language and identity see e.g. Fishman & Garcia (2010).

of L are also fully proficient in H, i.e. bilingual. Now consider a language policy that offers certain public services in L. There is no language barrier in public services for bilingual individuals, whether or not the policy is implemented. There is no simple communicative gain for them, since they speak H. Nonetheless, they might value the mere fact that their first language and with it their cultural heritage is recognized by the H dominated state. It has a symbolic value to speakers of L. Policies like using both languages H and L on bank notes are purely symbolic, but can still be valued by individuals due to the identity aspect of language.

Most language policies produce costs and benefits not only at the point in time they are first implemented, but every year they are in place.[3] Education policies are a good example here. Consider the introduction of an education program in a minority language L. There are set up costs, e.g. for the design of the program, teacher training, and production of textbooks. There are also significant variable costs, e.g. wages, teacher training and production of new text books. Concerning benefits, not only the L speaking pupils at the time of the implementation of the program benefit, but later pupils as well. This example shows that to properly evaluate language policies, present as well as future benefits have to be taken into account. In principle, standard cost-benefit analysis can account future effects of a policy by discounting (estimated) future costs and benefits and considering the *net present value* of the policy.

A number of language policy measures share the feature that their costs and benefits depend on the number of beneficiaries. In the case of the minority language education program, for example, the number of beneficiaries is related to the number of pupils speaking L. If the number of pupils increases, then, consequently, more pupils benefit from the policy and the costs of the policy increase. Roughly, one can say that costs and/or benefits of many language policies depend on the linguistic composition of the population subject to the policy. And this linguistic composition is by no means constant. Minority languages gain speakers, e.g. due to migration, or loose them, e.g. due to a general decline of the minority language. Hence, the estimation of future costs and benefits of a language policy should be based on estimates on changes in the linguistic composition of the population.

Largely unrelated to public economics and policy analysis, there is a literature on models that provide estimates or predictions of future changes in the linguistic composition.[4] They are called language competition or language dynamics models. Somewhat surprising, a large number of essays on language competition models were published in physics journals, which reflects the background of their authors. They apply models they normally use to describe physical processes to model language dynamics. Other models in the literature are inspired by models from biology. Several authors collaborated with scholars from linguistics and sociolinguistics to make the linguistic interpretation of their physics or biology models more sound. Some essays even show that model projections can fit empirical macro data on the linguistic composition.

[3]Some symbolic measures like adding a language to bank notes are an exception.

[4]See Section 3.1 for a more detail literature overview.

Another strand of literature analyzes language competition from an economics perspective in which one can find game theory approaches as well as voter models. What separates them from the non-economic models is a conception of individuals as utility maximizing actors. They investigate the evolution of the macro dynamics that stem from the individual decisions and identify (democratic) equilibria. In some models, the state plays an active role. In Kennedy & King (2005), for example, the state collects a tax and spends the revenue on language education policies. Individuals vote on the budget of the education program, and the state executes the democratic public will. In a cost-benefit analysis, on the other hand, the state is an instance that makes decisions from a macro perspective, weighing societal benefits against societal costs. Such a conceptualization is closer to actual democratic societies in which citizens elect political representatives who have to make decisions. As I outlined earlier, political decisions related to language are unavoidable and *laissez-faire* politics can lead to sub-optimal societal outcomes due to externalities.

In this thesis, I look at language policies and language dynamics from the perspective of the state or of policy makers. The question I want to answer is whether the (expected discounted) benefits of a specific language policy exceed its costs, and if so, by how much. An answer to this question yields an indication of whether a policy should be implemented or which of a list of possible policy options should be chosen. A combination of cost-benefit analysis and language competition models can provide such answers. As described earlier, language competition models produce projections for the future development of the linguistic composition of a population. These projections then inform estimates on future costs and benefits of a policy. Finally, these estimates derive a more realistic estimation of the net present value of a language policy. I call this approach *dynamic cost-benefit analysis*.

The underlying language competition model plays a crucial role in the dynamic cost-benefit analysis approach. It has to satisfy at least two conditions. First, it should be able to reproduce observed historical dynamics. If the model produces projections that do not coincide or, at least, show similarities with empirical observations, then estimates on future costs and benefits are basically worthless. Second, the model has to offer a meaningful way to model language policies. The straightforward way to include policies is to model them as a change in the model parameters. To do that, one has to have a clear interpretation of what a change in model parameters actually means in practice. Unfortunately, this is somewhat problematic in many of the physics and biology inspired model approaches, since parameters originally reflect physical properties of a physical system and not sociolinguistic characteristics of a society.

For this thesis, I start with a language competition model analyzed in Wickström (2005). Here, this model is referred to as model W. It is built on a conceptualization of individuals as rational actors. In the model, families weigh the communication and the identity aspect of language against one another when deciding which language(s) to transmit to their children. Throughout the thesis, I adapt and extend model W. I include more aspects of real life language dynamics such as migration and education, and present and analyze different functional forms of the general extended model.

This thesis consists of four self-contained essays. In the essay in Chapter 2,[5] my coauthors and I analyze whether revitalizing a declining minority language can be optimal in terms of welfare. To do so, we build on model W. We stay close to the original model, but use different functional expressions to model language transmission within the family context. I refer to this version of model W as model $\widetilde{W}$. As described above, families are utility maximizing actors who consider the communication value and the symbolic value of the languages in question. The former is represented by the number of speakers of a language, while the latter is modeled through the status of a language. The language dynamics are then driven by family formation and intergenerational language transmission. We analyze a scenario with two languages. Without state intervention, the minority language loses speakers and will disappear in the long run. The state has the opportunity to invest into status planning to increase the status of the minority language. Hence, the state can affect the decision making process of individual families. This investment is costly. At the same time, it can decrease the pace of the decline of the minority language or even stop it. We assume that the state wants to enable wide communication possibilities and to support the minority language. Both can be achieved by a high number of bilinguals. Hence, the objective of the state is to achieve a maximal amount of bilinguals at minimal costs. Combining the language competition model with this objective we get a dynamic control problem. We solve this problem and identify an optimal investment strategy. Last, we show that under certain conditions it can be optimal to invest sufficiently in the minority language for it to survive in the long run.

In the essay in Chapter 3,[6] I develop an extended version of model $\widetilde{W}$. In model W and model $\widetilde{W}$, the language dynamics are driven solely by family formation and language transmission from one generation to the next. Empirical and theoretical literature on language transmission, acquisition, and decline from sociology, sociolinguistics, as well as economics indicate that intergenerational language transmission is indeed one of the most important processes for language dynamics. If languages are not passed from one generation to the next, then they tend be lost after a few generations. This is, for example, well studied for migrant minority languages in the United States. First generation migrants use their heritage language at home and pass it on to the second generation. The second generation are bilingual and often don't pass their mother tongue to their children. Consequently, the third generation then is mostly monolingual in English. What the literature also shows is that intergenerational language transmission is not the only important process to be considered. A second pivotal process is language education. Languages taught at school, as medium of instruction or as foreign languages, affect the linguistic repertoires of pupils and hence the linguistic composition of the population as a whole. Moreover, language education is obviously interesting

[5]This essay was first published in *Mathematical Social Sciences*, cf. Templin *et al.* (2016).

[6]This essay was first published in *Rationality and Society*, cf. Templin (2018).

from a language policy perspective, since policies often determine which languages are used and taught in public schools. Language education is therefore modeled explicitly in the extended version of the model. The extended version, called model E^1, also takes into account concentration of speakers of the minority language, language learning by adults, and migration. Furthermore, I offer an operationalization of the abstract status parameter used in the previous models W and $\widetilde{W}$. After presenting and analyzing model E^1, it is applied to the case of Spanish and English in the United States. It is shown that model parameters can be estimated from quantitative empirical data and that the model can be fitted to observed historical Spanish-English language competition.

The extended model is used again in the essay in Chapter 4,[7] but in a streamlined version. This version is referred to as model E^1_{const}. In Chapter 3, language transmission and adult language learning are functional expressions of the relative status of the minority language as well as the linguistic composition. These functional expressions make use of parameters that reflect the relative importance of the status aspect with respect to the communication aspect of language in language related decisions. In contrast to many other model parameters, these parameters are not estimated from empirical data. Moreover, depending on the specific case as well as on the time span, the functional expressions remain relatively constant over time. Therefore, I offer in Chapter 4 a version of model E^1, in which the functional expressions are replaced by constant expressions. The advantage of the streamlined Model model E^1_{const} is that all its parameters can be estimated from empirical data. This is illustrated for the case of Spanish and Basque in the Basque Autonomous Communities in northern Spain. Despite the substitution of the more accurate functional expressions by constants, the streamlined model with estimated parameters can reproduce the observed historical language dynamics, at least for reasonable time horizons. At the end of the essay, I illustrate how the language competition model can be used to perform dynamic cost-benefit analysis.

In the first three chapters, I consider a situation with two languages (H and L) and three language groups: monolinguals in H, monolinguals in L and bilinguals. It is needless to say that the linguistic reality of most states in this world is more complex. Moreover, in the previous three chapters the age of individuals is not considered. When it comes to the formation of new families and the transmission of languages to young children, all individuals are implicitly assumed to be equally relevant. In a situation where a minority language is only spoken by the oldest generation, but not by young parents, this assumption is clearly problematic and most likely yields unrealistic projections. Therefore, I offer two additional extensions in Chapter 5. First, I propose and analyze an extension of the model from Chapter 3, that is model E^1, with multiple age groups, called model E^2_{mG}. Second, I consider the case of one majority language H and several minority languages $L_1, ..., L_n$. The model with just one minority language can then be seen as a special case of the more general model with multiple minority languages, called model E^2_{nL}. It is shown that on the one hand, both extensions are valuable especially for the analysis of policies that only target single minority language groups

[7]This essay is accepted for publication in *Language Problems and Language Planning*.

in a setting with multiple minority languages or of policies that only target certain age groups, like education policies. On the other hand, it is also shown that the simpler model E^1 with only one minority language and without age differentiation can yield a useful and easier to handle approximation of the more complex cases of model E^2_{mG} and model E^2_{nL}.

2 Optimal language policy for the preservation of a minority language **

2.1 Introduction

In many of the states in this world, one can find two or more larger language groups, often in form of a majority language and one or several minority languages. This is by no means a static situation, since "[a]ll over the world, people are stopping speaking minority languages and shifting to languages of wider communication" (Sallabank 2012, p. 104). This often results in the displacement of the minority languages by the majority language. To some extend such processes are inevitable and can be observed throughout human history. Nevertheless, in the modern world the decline of minority languages appears to occur much faster than ever before. It is predicted that 90 percent of the currently 7000 spoken languages will not survive the end of the century (Krauss 1992).

Language shift and maintenance

In response to this accelerated process of (minority) language decline, revitalizing and maintaining (endangered) minority languages is on the agenda of many of their speakers. Moreover, governments, non-governmental organizations as well as international organizations such as the European Union "are actively working to save and stabilize endangered languages" (Fernando *et al.* 2010, p. 49). In scientific discourses a large variety of arguments to support (minority) language rights or to save endangered languages were put forward over the past decades. In this essay we will not assess such arguments in detail or develop new ones,[8] but rather investigate in a formal model setting the possibilities, effects and costs of language policies aiming at saving endangered languages. To do so, we first have to identify causes of language shift as well as measures that are available to reverse language shift. Here again, we will not go into all the details and mostly refer to the extensive literature on this topics, see e.g. Fishman (1991), Crystal (2000), Nettle & Romaine (2000) and May (2011). Furthermore, we have to specify the target function: what is the desired state of affairs that language policies should aim at?

Referring to Nettle & Romaine (2000) and Crystal (2000), Sallabank groups different causes for language shift in four often overlapping main categories: a) natural catastrophes, famine, disease, b) war and genocide, c) overt repression and d) cultural/political/economic dominance, where the last one is the most common,

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[8]For an overview of the current discussions concerning language rights see e.g. May (2011) or Sallabank (2012). See also Fishman 1991 for a popular work on reversing language shift.

cf. Sallabank (2012, pp.103f). Since we are interested in such cases, where individuals *voluntarily choose* to change to the majority language or not to pass the minority language to the next generation, we concentrate on the last category. Especially in nation states with one official/national language (which is often but not necessarily the language of the majority) this language is dominant in education, politics, media and public life. In modern democratic states the result is "that the majority culture [...] is endemic and omnipresent; and minority cultures, having very little, if any, public legitimization and private space, thereby constantly decline in survival potential, the more their members participate in the 'greater general good'" (Fishman 1991, p. 63). Here, uneven power relations between the *national majority* and minorities play a major role. Minorities are often underrepresented in politics and in the public sphere and socially disadvantaged, cf. May (2011). This, in turn, can lead to negative attitudes towards the minority language, which are also internalized by its speakers (Sallabank 2012, p. 104). When the two main aspects of language are considered — language as a tool for communication and language as a carrier of cultural identity — it is no surprise, that a language that cannot be used in the majority of societal domains and that is furthermore stigmatized to some degree will not be learned, spoken or passed to the next generation.[9]

A language shift is a process that is typically comprised of three phases. In a first phase, called diglossia, formal language domains are dominated by the majority language which implies a loss of official and public functions of the minority language. This *forces* the speakers of the minority language to use the dominant one. In a second phase more and more speakers of the minority language become bilingual, while both languages are still used, at least in some domains. Especially among the younger generation one can observe a decreasing number of speakers. This causes a further decline of domains where the minority language can be or is used. The third phase finally is the replacement of the minority language: "For a generation or two, some bilingual arrangements may be observed, but often [...] these prove to be way-stations on the road to a new monolingualism in the larger language" (Edwards 2010, p. 6).

The language shift process can be counteracted by language policies aiming at the survival of the minority language. Language planning can be divided into three categories: status planning, corpus planning and acquisition planning. All three can have a positive impact on the chances of survival of minority languages. Through status planning, e.g. giving some official status to the minority language, the prestige of the language can be increased for its speakers as well as for the other members of the society. Corpus planning, which aims at standardizing the orthography and grammar of a language, can also increase its prestige and at the same time can reduce learning costs. Teaching the minority language at school, which belongs to the category of acquisition planning, enables students to learn the language properly/in the first place and can also have a positive impact on

[9]"The communicative value of languages is largely determined by the number of speakers it gives access to and by the status or social positions of these speakers" (Robichaud & Schutter 2012, p. 127).

its status and identity value. In general, (re)introducing and/or strengthening the minority language in at least some domains can enhance the chances that it stays vital.

In this essay we concentrate on the role of the state in language revitalization processes. We presuppose that the state is basically interested in supporting the minority language by guarantying minority language rights.[10] At the same time, we assume that the state aims at ensuring social cohesion by enabling wide communication possibilities. The existence of two linguistically segregated language groups can threaten the solidarity between the society members and hence social cohesion. Even without referring to a necessity of a shared national identity for solidarity and cohesion one can at least say that "a shared language contributes to democracy" (Robichaud & Schutter 2012, p. 135). Enabling wide communication possibilities while guarantying minority rights can be achieved through widespread bilingualism. If the minority language can be preserved in form of a relatively large number of bilingual individuals, the language minority is able to pass cultural values linked to the minority language to the next generations while communication possibilities throughout the society are assured. As outlined earlier, bilingualism is often a step towards the death of the minority language. Thus, preservation of a vital bilingual community requires a continuous effort by the state. In our model — and this is operationalized into the target function — the state tries to maximize the number of bilingual speakers at minimal expenditures.

Language competition models

In the past two decades a wide variety of language competition models were developed. One important point of departure for this new research on language competition was the work by Abrams & Strogatz (2003). There, a simple language competition model with two monolingual subpopulations is developed. The fraction of speakers of each language evolves according to a differential equation, which takes into account the size of the subpopulations and the prestige of both languages. Although the authors can fit their model to aggregated empirical data of endangered languages, it shows some weaknesses. In Abrams & Strogatz (2003) neither bilingual speakers nor the social structure of the population are considered. Moreover, it is predicted that always one of the two competing languages will die out in the long run. Due to such limitations, the model was revised and extended by many authors, especially from the field of (statistical) physics. Patriarca & Leppänen (2004) and Patriarca & Heinsalu (2009) include spatial components in their adaptions of the AS model. Taking geographical inhomogeneities into account they were able to show that it is possible that both languages survive in two

[10]As mentioned above, there are many arguments supporting such policies:
"Indeed, the dynamics of ethnic tension involving language, leading in some cases to political conflict, occur most often *not* when language compromises are made or language right are recognized, but where they have been historically avoided, suppressed or ignored" (May 2011, p. 161).
"So people's self-respect and dignity are often affected by the esteem their language gets from others or from the state. We might then justify different language policies by appealing to the importance of language recognition for individuals' dignity" (Robichaud & Schutter 2012, p. 136).

different geographical regions. Mira & Paredes (2005) introduce the concept of similarity between competing languages and prove that both languages can survive if they are close to each other. Stauffer *et al.* (2007) propose microscopic or individual based versions of the AS model and apply simulation techniques instead of averaging over the whole population. Mira & Paredes (2005), Minett & Wang (2008), Heinsalu *et al.* (2014) and others extend the A-S model by additionally considering bilinguals. Pinasco & Romanelli (2006) propose a Lotka-Volterra type model inspired by population dynamics to model language competition and also show the possibility of coexistence. Spatial extensions of this model can be found in Kandler & Steele (2008) and Kandler *et al.* (2010). A good review of the different approaches is given in Patriarca *et al.* (2012).

In the model of Abrams and Stogatz (A-S model) speakers of two language H and L are assumed. Speakers of H can convert to speakers of language L and vice versa, while the population size remains constant. Minett and Wang point out that "in practice, [...] typically a speaker does not suddenly give up one language completely in favor of an other" (Minett & Wang 2008, p. 23). Therefore, they include bilingual speakers in their adoption of the A-S model. Furthermore, Abrams and Strogatz implicitly consider language transmission from one generation to the other when fitting their mathematical model to empirical data from more than a hundred years without theorizing this fact. Minett and Wang therefore consider two modes of language transmission: 1) vertical, i.e. transmission from parents to their children and 2) horizontal, i.e. (adults) learning the second language and becoming bilingual. For the vertical mode, a uniparental model of transmission is applied. In contrast, Wickström (2005) only considers vertical transmission, but explicitly models family formation. It is assumed that adults mate due to a random search and matching process with a success probability that is smaller for couples with an H-monolingual and a L-monolingual partner than for all the other possible couples. In the so formed families offspring is produced and raised in one — or in some cases both — of the parents' languages, depending on the communicational value of each language and their status/prestige. As Wickström (2005) we only consider the vertical mode, i.e. intergenerational language transmission[11].

In Wickström (2014) it is illustrated that the A-S model and its extension by Minett & Wang (2008) can be reformulated in terms of the general model presented in Wickström (2005). Furthermore the spatial model in Patriarca & Leppänen (2004) can be interpreted as a version of the Wickström framework with two subpopulations I and II, who value languages H and L differently. It is shown that under some general assumptions on the nexus between transition probabilities and the size of the subpopulations stable steady states of the system are the same as derived by Patriarca & Leppänen (2004) in spatial terms. For this essay we build on the general model formulation presented in Wickström (2005) and Wickström (2014). Hence we consider speakers of a high status majority language H, speakers of a low status minority language L and bilingual speakers B.

[11]Transmission in the family is the 'gold standard' of language vitality and the most important factor in language survival (Fishman 1991, p. 113).

Only some of the language dynamics models outlined above deal with language revitalization policies. In terms of a mathematical model, such policies can be operationalized as a change of relevant model parameters that are related to the linguistic environment: "political, social and/or economic changes can lead to a change in the sociolinguistic environment and consequently to a change in the competition dynamics" (Kandler *et al.* 2010, p. 3859f). Yet, most often model parameters are assumed to be constant over time. To maintain a bilingual equilibrium Minett & Wang (2008) suggest a simple intervention strategy: whenever the amount of speakers of the minority language drops below some threshold value, then the status of the minority language or some other model parameters have to be increased. That such a "dramatic intervention" (Fernando *et al.* 2010, p. 51) is quite unrealistic, was already mentioned in Minett & Wang (2008). It can be seen as a theoretical approximation of a more sophisticated intervention, which starts to increase the minority language status when the numbers come close the threshold.

A greater effort to model language planning was undertaken in Fernando *et al.* (2010). They consider intergenerational language transmission as well as horizontal transmission. In contrast to Wickström (2005) parents do not just choose one or two languages to raise their children in. Instead, the probability that a child speaks a language l strongly depends on the amount of l-conversations it is exposed to. Within the family this amount only depends on the linguistic repertoires of the parents. Furthermore, Fernando *et al.* consider the influence of the community by taking into account conversations heard in the public sphere and languages taught at school. This is also reflected in three different kinds of interventions contemplated there: 1) increasing the status of the minority language,[12] 2) increase the amount of the minority language heard in public and 3) formal language teaching. In their simulations Fernando *et al.* illustrate the effect of different kind of governmental interventions.

After 100 years simultaneously the status of the minority language as well as the amount of that language used in public are increased and the minority language is taught in formal education to some monolinguals of the high-status language. In the model this is realized by increasing three corresponding parameters at year 100. Citing Fernando *et al.* (2010, p. 51) when reviewing Minett & Wang (2008) one may ask: "How such a dramatic intervention could be achieved is not explained".

In Kandler *et al.* (2010) the authors fit their basic model with time-independent parameters ("shift coefficients") to data on language competition between Welsh

[12]Unlike most of the models listed above, there is no explicit status parameter in Fernando *et al.* (2010). The status of the minority language is reflected by the parameter that "measures the effectiveness of hearing language [the minority language] in motivating its learning (i.e. the receptiveness of the child to [the minority language])" in an HH or HB family (p. 60). This parameter is not to be understood as an individual trait of the child. Among other things, it represents "the "status" of [the minority language], where status is used to mean the entire constellation of societal factors that motivate the learning of a given language" (p. 60, emphasis in original). This status related parameter functions as an amplifier for L-conversation heard by a child.

and English in Wales. For the period from 1901 to 1971 the model captures the observed dynamics quite well. Yet, the basic model could not adequately account for maintenance interventions implemented in the past 40 years, which could be the cause of reduced decline of Welsh. Therefore, the authors extend their basic model "by incorporating a simplified concept of (extended) diglossia" (p. 3862). The high-status language is used in important domains as higher education or non-local businesses. This yields an incentive for speakers of the minority language to become bilingual. At the same time, political interventions might support the low status language in other domains such as local legislation. This, can create incentives for monolinguals of the dominant language to become bilingual and for bilingual parents to transmit both languages to the next generation. Kandler *et al.* introduce an additional term in their model that captures the demand of participation in domains where the low status language is used. This demand is reflected by the parameter w_1. Assuming that w_1 doubles after 1971, the extended model is able to approximate the empirical data. The increase of w_1 is a result of language planning incentives.

In the above three examples, language planning policies are modeled as a change in model parameters. These changes occur at some single point in time, i.e. at some point in time the value of a parameter (or multiple parameters) jumps to another value. Depending on the parameter that is changed as well as on the size of the jump, such a "dramatic intervention" might be rather unrealistic. In their adoption of the model proposed in Minett & Wang (2008), Bernard & Martin (2012) also include the opportunity for policy makers to alter the status of the minority language. In contrast to the previous approaches, they assume that the variation of the status at each time step is bounded. Hence, the size of the jump is limited, which yields a potentially more realistic model for intervention. Setting up a dynamic control model, they were able to show that when starting in a given domain there exist adequate intervention strategies such that both monolingual subpopulations can be preserved.

In this essay we also propose a language competition model with dynamic intervention. A first difference to the model analyzed in Bernard & Martin (2012) is that we build on the general model formulation presented in Wickström (2005). Secondly, in our approach the status can not be regulated directly. Instead, we assume that the state has a certain budget that can be used for status planning. To increase or even stabilize the status of a (minority) language continuous investments into status planning are necessary. Hence, we assume that whenever the state reduces its efforts to maintain the minority language beyond a certain value, then the status of that language decreases. This implies that without any intervention the status tends to zero in the long run. The investment strategy is denoted by a process $(s_t)_{t\geq 0}$. Since the budget is assumed to be finite, we can normalize the investment such that $s_t \in [0, 1]$. Thirdly, we propose an optimal control model. The aim of languages policies is not to maintain monolingual subpopulations of both languages, but to maintain both languages in a scenario with large communication possibilities throughout the society. Hence, the aim is to maximize the amount of bilingual speakers. Furthermore, investments into status planning are costly. Therefore, the objective here is to maximize the bilingual subpopulation at minimal costs.

The dynamic control model proposed below is a three-state system. The three states are: the fraction of speakers of language H (denoted by X_H), the fraction of speakers of language L (denoted by X_L) and the relative status of language L (denoted by S). The fraction of bilingual speakers is simply given by $X_B = 1 - X_H - X_L$, and the relative status of the majority language H is given by $1 - S$. In Fernando *et al.* (2010) the authors criticize such an assumption in the model of Minett and Wang because it implies "that it is impossible to make one language more attractive without making the other less so" (Minett & Wang 2008, p. 50). However, in a language competition situation, where individuals have to decide for one language, the other or both, this assumption makes sense when we think of relative attractiveness instead of absolute attractiveness. Hence, instead of statements as 'language H has an attractiveness value of 3.5' the model here only allows statements like 'language H is three times as attractive as language L'.

The evolution of the system is described by three differential equations. The status can be affected by state intervention s, i.e. $\dot{S} = g(s, S)$, where g is some function increasing in s. The evolution of the distribution of language repertoire groups R depends only on the distribution itself and on the status S. Hence, the fractions X_R, $R = H, L, B$, can be influenced by state intervention, but only indirectly through the controlled status.

This essay is organized as follows. In Section 2.2 the general language dynamic model is introduced. In Section 2.3 we suggest specific functional forms for the general model described in the previous section. Section 2.4 aims at identifying the optimal public investment strategy. Furthermore, some general statements on steady states of the optimally controlled system are derived. In Section 2.5 we consider some case studies to illustrate our results numerically. Section 2.6 provides conclusions and some remarks for future research.

2.2 Model

We consider a (large) population consisting of individuals equipped with one of three different language repertoires R: monolingual speakers of the dominant language H, monolingual speakers of the minority language L and bilinguals speakers B. The relative sizes (fractions of the population) of the respective language repertoire groups are denoted by X_H, X_L and X_B. The fractions add up to 1, hence $X_B = 1 - X_H - X_L$. The variable S represents the relative status of the minority language L in the society.

2.2.1 Family formation

In every generation individuals form families. There are six family types F: HH (two H monolinguals), HL, HB, LL, LB and BB. Family formation is assumed

$\mathbf{F}$	$\psi_{\mathbf{F}}$
HH	$X_H^2 + X_H X_L$
HL	0
HB	$2X_H X_B$
LL	$X_L^2 + X_H X_L$
LB	$2X_L X_B$
BB	X_B^2

Table 1: Distribution of families for a given distribution of adult speakers.

to be random but restricted by the condition that both adults should share a common language, i.e. they should be able to communicate with each other. Hence, couples with an H-monoglot and a L-monoglot are excluded. Given any distribution of speakers X_H, X_L, X_B, the expected distribution of family types is given in Table 1, where ψ_F denotes the fraction of F-type families.[13]

2.2.2 Family behavior

Families bring up their children either as monolinguals in H or L, or as bilinguals. The fraction of F-type families bringing up children with language repertoire R is denoted by $q_R(F;\cdot) \in [0,1]$. Naturally, the q's add up to one: for every family type F

$$\sum_R q_R(F;\cdot) \;=\; 1.$$

The $q-$functions are one of the main ingredients of the model proposed here. Parents choose a language repertoire depending on their own languages, on their emotional attachment to those languages as well as on the communication values of all the languages at hand. Therefore, the fraction of families of type F raising their children as R's, $R = H, L, B$, varies with the current distribution of speakers in the society as well as with the status of languages H and L. Hence, $q_R(F;\cdot) = q_R(F; X_H, X_L, S)$. The dependence on the variables X_H, X_L captures the practical advantage of belonging to a certain language group, since they measure the frequency with which an individual encounters another individual in group H, L and B, respectively, and hence measure how many people one can communicate with. Following the individual utility maximization approach developed in Wickström (2005), we assume that q_H is non-decreasing in X_H, and

[13]See Section 5.2.2 for the derivation of the expected distribution ψ_F in a more general case.

non-increasing in X_L, and vice versa for q_L:

$$\frac{\partial q_H(F;X_H,X_L,S)}{\partial X_H}, \frac{\partial q_L(F;X_H,X_L,S)}{\partial X_L} \geq 0,$$
$$\frac{\partial q_B(F;X_H,X_L,S)}{\partial X_H}, \frac{\partial q_B(F;X_H,X_L,S)}{\partial X_L} \geq 0,$$
$$\frac{\partial q_H(F;X_H,X_L,S)}{\partial X_L}, \frac{\partial q_L(F;X_H,X_L,S)}{\partial X_H} \leq 0.$$

This reflects the first aspect of language mentioned in the introduction: language as a tool for communication. The second aspect – language as a carrier for cultural identity – is reflected in the dependence of the q's on the family type F and the relative status of the minority language S. It is hypothesized that the emotional attachment in the family to a certain language, and hence the frequency of its transmission to the next generation, depends on its strength in the family. The stronger the position of a language l in the family, the higher is the fraction q_l:

$$1 \geq q_H(HH;\cdot) \geq q_H(HB;\cdot) \geq q_H(BB;\cdot) \geq q_H(LB;\cdot) \geq q_H(LL;\cdot) \geq 0,$$
$$0 \leq q_L(HH;\cdot) \leq q_L(HB;\cdot) \leq q_L(BB;\cdot) \leq q_L(LB;\cdot) \leq q_L(LL;\cdot) \leq 1.$$

It is furthermore assumed that both parents shall be able to communicate with their children, cf. Fernando *et al.* (2010). Hence,

$$q_H(LB;\cdot) = q_H(LL;\cdot) = 0,$$
$$q_L(HB;\cdot) = q_L(HH;\cdot) = 0.$$

The average emotional attachment to a language l also depends on the general prestige or cultural status of the languaga in the society. The higher the status, the higher is the willingness of its speakers to pass their language to the next generation. We therefore assume that q_H is non-increasing in S, while q_L is non-decreasing in S:

$$\frac{\partial q_H(F;X_H,X_L,S)}{\partial S} \leq 0,$$
$$\frac{\partial q_L(F;X_H,X_L,S)}{\partial S} \geq 0.$$

From the assumptions made above two properties of the q functions can be concluded. Since $q_L(HH)$ and $q_H(LL)$ are equal to zero, we get

$$\frac{\partial q_H(HH;X_H,X_L,S)}{\partial X_H} = \frac{\partial q_L(LL;X_H,X_L,S)}{\partial X_L} = 0.$$

Furthermore, $q_L(HB;\cdot) = q_H(LB;\cdot) = 0$ yield

$$\frac{\partial q_H(HB;X_H,X_L,S)}{\partial X_H} = \frac{\partial q_L(LB;X_H,X_L,S)}{\partial X_L} = 0.$$

2.2.3 Dynamics

While in Abrams & Strogatz (2003) a constant population size is assumed, other researches explicitly model *logistic* population growth, see e.g. Pinasco & Romanelli (2006) or Kandler *et al.* (2010). If growth rates and carrying capacities vary between the language repertoire groups, then the population dynamics can have a major impact on possible steady states. Yet, if growth is homogeneous throughout all the groups and a common carrying capacity is assumed, then population dynamics do not affect the steady states, cf. Heinsalu *et al.* (2014). In this essay we also assume homogeneous growth at rate θ and a common carrying capacity K. Since the number of children born in a family and thus the overall population dynamics is independent of the status[14], considering status planning does not violate the homogeneity assumption. Therefore, the model proposed here "could describe the interaction between linguistic groups that have already reached a state in which reproduction and access to resources takes place in similar ways" (Heinsalu *et al.* 2014, p. 5), and can not account for situations in which one language repertoire group has much less access to resources than the other language repertoire groups.

Let N denote the size of the population, and N_R, $R = H, L, B$, denote the sizes of the language repertoire groups. The dynamics of the overall population size is described by the logistic differential equation

$$\dot{N} = \theta N \left(1 - \frac{N}{K}\right) = \theta N \sum_R \left(\left[\sum_F q_R(F; \cdot)\psi_F \right] - \frac{1}{K} N_R \right).$$

The size of language repertoire group R changes according to

$$\dot{N}_R = \theta N \sum_F q_R(F; \cdot)\psi_F - \theta \frac{N}{K} N_R.$$

Therefore, the relative size of language repertoire group R, $X_R = N_R/N$, evolves according to

$$\dot{X}_R = \theta \left(\sum_F q_R(F; \cdot)\psi_F - X_R \right).$$

For languages H and L this reads as

$$\frac{\dot{X}_H}{\theta} = (X_H^2 + X_H X_L) q_H(HH) + 2X_H X_B q_H(HB) + X_B^2 q_H(BB) - X_H, \tag{2.1}$$

$$\frac{\dot{X}_L}{\theta} = (X_L^2 + X_H X_L) q_L(LL) + 2X_L X_B q_L(LB) + X_B^2 q_L(BB) - X_L, \tag{2.2}$$

where $q_l(F) = q_l(F; X_H, X_L, S)$, $l = H, L$.

[14] The relative status S only influences parents decisions on the language repertoires of their children.

The status variable

The status of the minority language L is expressed in the variable S, $0 \le S \le 1$. Investments in status planning s can increase the status of the minority language:

$$\dot{S} = f(S, s) - \mu S. \tag{2.3}$$

It is assumed that the function f is non-increasing in S and non-decreasing in s. Furthermore, for $s = 0$ the function f should be zero. This implies, that without any state intervention the relative status of the minority language B converges to zero at rate μ.

2.2.4 The objective function

The aim of state intervention is a large bilingual subpopulation. At the same time, state interventions to increase the status of the minority language are costly. Hence, the decision maker is looking for an investment policy $(s(t))_{t\geq 0}$, $s_t \in [0, 1]$, that yields a high level of individual bilingualism (benefit) at low costs. By $w(X_H(t), X_L(t), s(t))$ we denote the value of the system at time t, i.e. benefits minus costs at time t. We require w to be increasing in $X_B = 1 - X_H - X_L$, non-increasing in X_H and X_L, and decreasing in s. The total discounted value is given by

$$\int_0^\infty e^{-rt} w(X_H(t), X_L(t), s(t)) dt,$$

where $r \in (0, 1)$ denotes the discount rate. The problem of finding the best investment strategy for language maintenance can now be formulated as a maximization problem:

$$\max_{(s_t)_{t\geq 0}} \int_0^\infty e^{-rt} w(X_H(t), X_L(t), s(t)) dt.$$

Note, that $S(t)$ and therefore $X_H(t)$ and $X_L(t)$ depend on the size of s prior to time t, cf. (2.3), (2.1) and (2.2).

2.3 Specific functional forms

In this section we provide specifications of the q-functions, the status dynamics and the objective function that satisfy the general assumptions made above.

For parameters $0 \le \eta < \beta < \delta$ and $\varepsilon + \gamma < \zeta < 1$ let

$$\begin{aligned}
q_H(HH; X_H, X_L, S) &= 1 - \eta S X_L, \\
q_H(HB; X_H, X_L, S) &= \max\{0, \zeta(1 - S) - \beta S X_L\}, \\
q_H(BB; X_H, X_L, S) &= \max\{0, \varepsilon(1 - S) + \gamma(1 - S) X_H - \delta S X_L\},
\end{aligned}$$

and

$$q_L(LL; X_H, X_L, S) = 1 - \eta(1-S)X_H,$$
$$q_L(LB; X_H, X_L, S) = \max\{0, \zeta S - \beta(1-S)X_H\},$$
$$q_L(BB; X_H, X_L, S) = \max\{0, \varepsilon S + \gamma S X_L - \delta(1-S)X_H\}.$$

These constructions imply, that given a sufficiently high fraction of H speakers in the society and a sufficiently low status of the minority language L, bilingual or even mixed couples (LB) will not raise their children as monolinguals in L, since in this scenario neither L is a very useful communication tool in this society nor can the prestige of this language really compensate the communication disadvantage.

Throughout the essay we will assume η to be zero. In this case the system dynamics simplify to

$$\frac{\dot{X}_H}{\theta} = X_B\left[2X_H q_H(HB; X_H, X_L, S) + X_B q_H(BB; X_H, X_L, S) - X_H\right], \quad (2.4)$$
$$\frac{\dot{X}_L}{\theta} = X_B\left[2X_L q_L(LB; X_H, X_L, S) + X_B q_L(BB; X_H, X_L, S) - X_L\right]. \quad (2.5)$$

2.3.1 Dynamics for fixed status

For the moment let S be fixed. The essential dynamics of X_H and X_L can each be described by two parameters, cf. Wickström (2005). These parameters are introduced in the following. Let $X_L^{\Delta}(S)$ denote the fraction of L speakers where $X_H = 0$ and $\dot{X}_H = 0$. Hence,

$$q_H(BB; X_H, X_L, S) = 0 \Rightarrow \varepsilon(1-S) - \delta S X_L = 0 \Leftrightarrow X_L^{\Delta}(S) = \frac{\varepsilon}{\delta}\frac{1-S}{S}.$$

For X_H^{Δ} respectively we get

$$X_H^{\Delta}(S) = \frac{\varepsilon}{\delta}\frac{S}{1-S}.$$

Next we look for X_H^* and X_L^*. X_H^* is the fraction when $\dot{X}_H = 0$ given $X_L = 0$. Hence, X_H^* is a solution to

$$0 = 2X_H q_H(HB; X_H, X_L, S) + (1-X_H) q_H(BB; X_H, X_L, S) - X_H,$$

or, with the above specifications, the unique positive solution of the quadratic equation

$$0 = \gamma X_H^2 - \left[2\zeta + \gamma - \varepsilon - \frac{1}{1-S}\right] X_H - \varepsilon. \quad (2.6)$$

Note, $X_H^* < 1$ iff $S > 1/2\zeta$. From this, we easily conclude that X_H^* is increasing in ζ, ε and γ, and decreases with an increase of S. On the other hand, X_H^{Δ} increases in ε and S and decreases with an increase in γ. It is unaffected by a change of ζ.

From the relations between X_H^{Δ}, X_L^{Δ} and X_H^*, X_L^* we can identify possible bilingual equilibria for the fixed status S:

Lemma 2.3.1.1. *Let $\eta = 0$.*

(a) If $X_H^\Delta \leq X_H^* < 1$ there exists a stable equilibrium with $0 < X_H < 1$ and $X_L = 0$; the fraction of H-speakers equals X_H^*

(b) If $X_L^\Delta \leq X_L^* < 1$ there exists a stable equilibrium with $0 < X_L < 1$ and $X_H = 0$; the fraction of L-speakers equals X_L^*

(c) If $1 \geq X_H^\Delta > X_H^*$ and $1 \geq X_L^\Delta > X_L^*$, we have a stable equilibrium with bilinguals and monolinguals in both languages ($X_H, X_L, X_B > 0$).

Lemma 2.3.1.2. *Let $\eta = 0$. For monolingual stable equilibria the following statements hold true*

(a) $X_H = 1$ is a stable equilibrium if and only if $S \leq 1 - 1/2\zeta$.

(b) $X_L = 1$ is not a stable equilibrium

(c) $X_H, X_L \in (0, 1)$ with $X_H + X_L = 1$ is stable iff

$$X_H q_H(HB; X_H, X_L, S) + X_L q_L(LB; X_H, X_L, S) \geq \frac{1}{2}. \tag{2.7}$$

A necessary condition for this last inequality is $S \leq 1 - 1/2\zeta$.

Lemma 2.3.1.1 can be established using a phase diagram, cf. Wickström (2005). The proof of Lemma 2.3.1.2 is found in the Appendix.

2.3.2 Variable status and status control

Now we specify the dynamics of the minority language status S, which is increasing as a result of investments into language policies and decreasing due to a general negative trend. We assume the following functional form:

$$\dot{S} = f(S, s) - \mu S = \nu(1 - 2S)\sqrt{s} - \mu S, \tag{2.8}$$

where $\nu > 0$ is a model parameter correlated to the effectiveness of intervention. Here two assumptions are made: a) for a low status language the necessary effort to increase its status is low, while for a high status language it takes more effort. b) Language L stays the minority language. This assumptions is expressed in the term $(1 - 2S)$. The status can not exceed 1/2, while the $(1 - S)$, which can be interpreted as the status of H, does not fall below 1/2. H can be thought as the first official language.

The control variable s is bounded ($s \leq 1$). Thus, any steady state status S ($\dot{S} = 0$) has an upper bound:

$$S \leq \frac{\nu}{2\nu + \mu}.$$

Since X_H^* is decreasing in S, while X_H^Δ increases in S, Lemma 2.3.1.1 (a) yields a second upper bound for S, which is relevant for equilibria with $0 < X_H < 1$ and $X_L = 0$. A third one results from Lemma 2.3.1.1 (b), see below. A minimal value for this kind of equilibrium is given by $X_H^*(S) < 1$, where X_H^* is the unique positive solution to (2.6).

We therefore introduce the following status thresholds

$$\begin{aligned} \overline{S} &:= \frac{\nu}{2\nu + \mu}, \\ \widetilde{S} &: X_H^*(\widetilde{S}) = X_H^\Delta(\widetilde{S}), \\ \underline{S} &:= 1 - \frac{1}{2\zeta}. \end{aligned}$$

Note, due to symmetry it holds $X_L^*(1 - \widetilde{S}) = X_L^\Delta(1 - \widetilde{S})$. Table 2 shows possible stable equilibria for the problem with fixed a status corresponding to these threshold values. The second interval is not empty if $\varepsilon/\delta \leq 1/2\zeta$. The third interval is not empty if $\varepsilon/\delta \geq 1 - 1/2\zeta$. Figure 1 illustrates some of the cases listed in Table 2.

$S \in$	$[0\,,\,\underline{S}]$	$(\underline{S} \vee 1 - \widetilde{S}\,,\,\widetilde{S}]$	$(\underline{S} \vee \widetilde{S}\,,\,1 - \widetilde{S})$
equil.	H, HL	HB, LB	HLB

Table 2: Possible stable equilibria for the fixed status problem for different values of S.

The first line contains intervals for S, while the second one shows the corresponding potential stable equilibria. "H, HL" means that a pure H-monolingual steady state as well as a steady state with monolingual speakers of H and L is possible.

To find optimal state intervention strategies we need to consider the derivatives of the function $f(S, s) = \nu(1 - 2S)\sqrt{s}$:

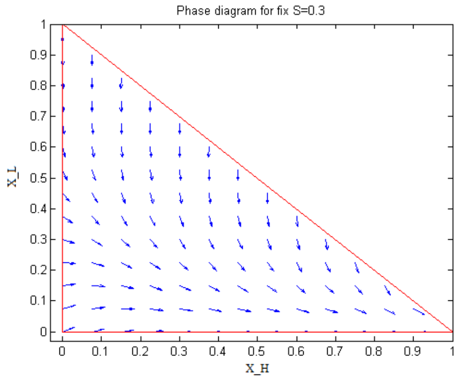
$$\frac{\partial f}{\partial s}(S, s) = \frac{\nu}{2}\frac{1 - 2S}{\sqrt{s}}, \tag{2.9}$$

$$\frac{\partial f}{\partial S}(S, s) = -2\nu\sqrt{s}. \tag{2.10}$$

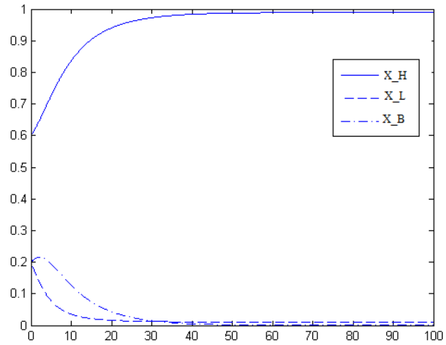
2.3.3 Objective

Departing at the initial state $X_H(0), X_L(0)$ and $S(0)$ the aim of the optimization problem is to find the best investment policy $(s(t))_{t\geq 0}$ such that, $r \in (0, 1)$, $k > 0$, $\xi \in [0, 1]$,

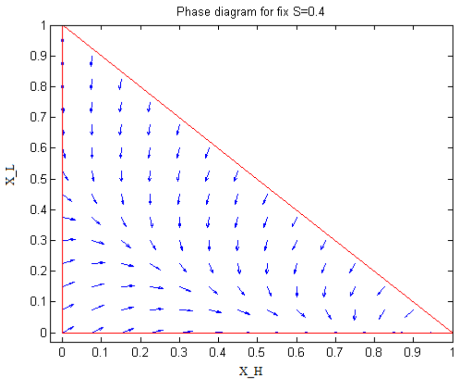
$$\int_0^\infty e^{-rt}\left(k \cdot X_B(t) - [X_L(t) + X_B(t)]^\xi s(t)\right) dt \tag{2.11}$$



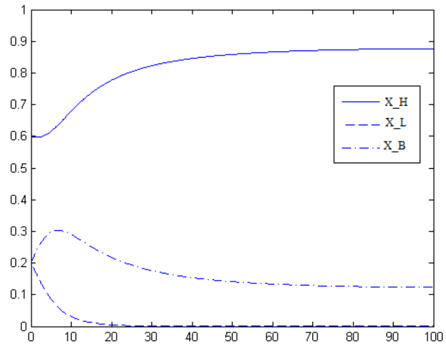
(a) $S = 0.3 < 0.375 = \underline{S}$



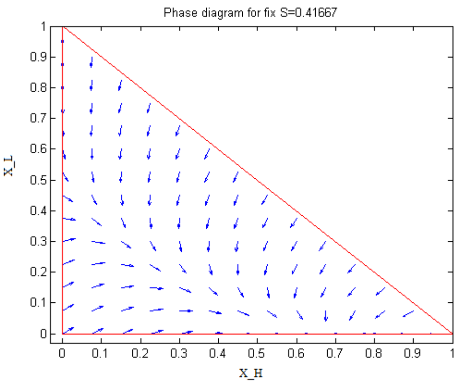
(b) $S = 0.3 < 0.375 = \underline{S}$



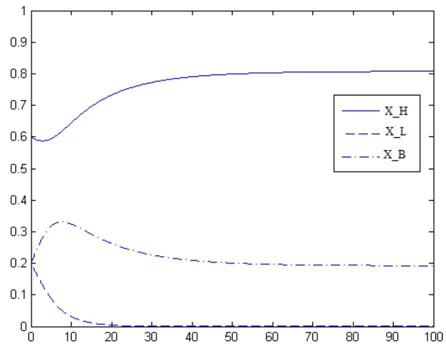
(c) $S = 0.4 < 0.49 \approx \min\{\widetilde{S}, 1 - \widetilde{S}\}$



(d) $S = 0.4 < 0.49 \approx \min\{\widetilde{S}, 1 - \widetilde{S}\}$



(e) $S = \overline{S} \approx 0.42$



(f) $S = \overline{S} \approx 0.42$

Figure 1: Phase diagrams and trajectories for fixed S.
Phase diagrams and trajectories for fixed S for different values of S. For the trajectories the initial distribution is $X_H = 0.6$ and $X_L = 0.2$. Parameters are as in Example 2.5.0.1 in Section 2.5.

is maximized, while the system is developing according to (2.4), (2.5) and (2.8). For $\xi = 0$ the costs for the state intervention do not depend on the numbers of speakers of language L. Here one can think of adding language L to (street-)signs. For $\xi = 1$ the costs linearly increase with the number of speakers - one could think of bilingual education in schools.

2.4 Optimal control and optimal steady states

Substituting $X_L + X_B$ by $1 - X_H$ in the objective function, the Hamiltonian can be expressed as

$$\mathcal{H}(X_H, X_L, S, s) = k \cdot X_B - (1 - X_H)^{\xi} s + \lambda_H \dot{X}_H + \lambda_L \dot{X}_L + \lambda_S \left(f(S, s) - \mu S \right), \tag{2.12}$$

where λ_H, λ_L and λ_S are the costate variables measuring the marginal value of the corresponding state variables X_H, X_L and S, respectively.

We assumed that the control variable is bounded, i.e. that the budget for language policies fostering bilingualism is limited. This budget constraint is formalized by the inequality $s \leq 1$. To include the constraint in the formal model we define the Lagrangian $\mathcal{L} := \mathcal{H} + \omega(1 - s)$, where ω is the Lagrange multiplier. For the identification of the optimal intervention at a given state we consider the derivative of $\mathcal{L}$ with respect to the control variable s:

$$\mathcal{L}_s = -(1 - X_H)^{\xi} + \lambda_S \frac{\partial f(S, s)}{\partial s} - \omega. \tag{2.13}$$

To identify an optimal intervention, we are looking for s and ω such that $\mathcal{L}_s = 0$ and $\omega(1 - s) = 0$. We have

$$\mathcal{L}_s = 0 \;\Leftrightarrow\; (1 - X_H)^{\xi} + \omega = \lambda_S \cdot \underbrace{\frac{\partial f(S, s)}{\partial s}}_{\geq 0} \quad \Rightarrow \quad \lambda_S \geq 0.$$

Note, if $X_H < 1$ then we even have $\lambda_S > 0$. For the explicit form of the function f defined in (2.8) we get

$$\begin{aligned} \mathcal{L}_s = 0 \Leftrightarrow (1 - X_H)^{\xi} + \omega &= \lambda_S \cdot \frac{\nu}{2} \frac{1 - 2S}{\sqrt{s^*}} \\ \Leftrightarrow s^* &= \left(\lambda_S \frac{\nu}{2} \frac{1 - 2S}{(1 - X_H)^{\xi} + \omega} \right)^2 . \end{aligned} \tag{2.14}$$

The second derivative of $\mathcal{L}$ with respect to the control variable s is non-positive if $\lambda_S > 0$ in which case the Legendre Clebsch condition is satisfied. Whenever $X_H = 1$, in which case $\lambda_S = 0$ could be possible, $s = 0$ is obviously optimal. Applying the optimal control we have

$$\dot{S} \;=\; f(S, s^*) - \mu S \;=\; \lambda_S \frac{\nu^2}{2} \frac{(1 - 2S)^2}{(1 - X_H)^{\xi} + \omega} - \mu S. \tag{2.15}$$

If the constraint is inactive, i.e. $s < 1$, then $\omega = 0$. If, in contrast, the constraint is active ($s = 1$), then

$$\omega = \lambda_S \frac{\nu}{2}(1 - 2S) - (1 - X_H)^{\xi} \geq 0. \tag{2.16}$$

2.4.1 Stationary points

To state the co-state equations we first introduce some functions. For $l = H, L$ set

$$g_l(X_H, X_L, S) := 2X_l q_l(lB; X_H, X_L, S) + X_B q_l(BB; X_H, X_L, S) - X_l,$$

which equals $\dot{X}_l/(\theta X_B)$ whenever $X_B > 0$. Then,

$$\mathcal{H} = X_B(k + \theta\lambda_H g_H + \theta\lambda_L g_L) - (1 - X_H)^\xi s + \lambda_S(f(S, s) - \mu S).$$

Using this notation we have

$$\begin{aligned}
\mathcal{H}_{X_H} &= -(k + \theta\lambda_H g_H + \theta\lambda_L g_L) \;+\; \theta\lambda_H X_B \frac{\partial g_H}{\partial X_H} \;+\; \theta\lambda_L X_B \frac{\partial g_L}{\partial X_H} \\
&\quad + \frac{\xi}{(1 - X_H)^{1-\xi}} s, && (2.17)\\
\mathcal{H}_{X_L} &= -(k + \theta\lambda_H g_H + \theta\lambda_L g_L) \;+\; \theta\lambda_H X_B \frac{\partial g_H}{\partial X_L} \;+\; \theta\lambda_L X_B \frac{\partial g_L}{\partial X_L}, && (2.18)\\
\mathcal{H}_S &= \theta X_B \left(\lambda_H \frac{\partial g_H}{\partial S} + \lambda_L \frac{\partial g_L}{\partial S}\right) + \lambda_S \left(\frac{\partial f(S, s)}{\partial S} - \mu\right). && (2.19)
\end{aligned}$$

The co-state equations are then given by

$$\begin{aligned}
\dot{\lambda}_H &= r\lambda_H - \mathcal{H}_{X_H},\\
\dot{\lambda}_L &= r\lambda_L - \mathcal{H}_{X_L},\\
\dot{\lambda}_S &= r\lambda_S - \mathcal{H}_S.
\end{aligned}$$

To find inner stationary points we try to identify solutions

$$(\widehat{X}_H, \widehat{X}_L, \widehat{S}, \widehat{\lambda}_H, \widehat{\lambda}_L, \widehat{\lambda}_S)$$

to

$$0 = \dot{X}_H = \dot{X}_L = \dot{S} = \dot{\lambda}_H = \dot{\lambda}_L = \dot{\lambda}_S.$$

For $\widehat{X}_H$ and $\widehat{X}_L$ to be stationary we need either $\widehat{X}_B = 0$ or $g_H(\widehat{X}_H, \widehat{X}_L, \widehat{S}) = g_L(\widehat{X}_H, \widehat{X}_L, \widehat{S}) = 0$.

Note, any steady state status $0 < \widehat{S} < \overline{S}$ corresponds to a steady state control variable $0 < \widehat{s}^* < 1$ and hence to some $\widehat{\omega} = 0$. In this case, the stationarity of the status ($\dot{S}(\widehat{S}, \widehat{\lambda}_S) = 0$) yields an explicit relation between $\widehat{S}$ and $\widehat{\lambda}_S$, cf. (2.15):

$$\widehat{\lambda}_S = \frac{2\mu}{\nu^2} \frac{\widehat{S}}{(1 - 2\widehat{S})^2} (1 - \widehat{X}_H)^\xi. \tag{2.20}$$

Plugging this into (2.14) we get for the stationary optimal intervention

$$\widehat{s}^* = \left(\frac{\mu}{\nu} \frac{\widehat{S}}{1 - 2\widehat{S}}\right)^2 < 1. \tag{2.21}$$

If $\widehat{S} = \overline{S}$, then $\widehat{s}$ has to be equal to one and thus $\widehat{\lambda}_S \geq 2\frac{2\nu+\mu}{\nu\mu}(1 - X_H^*(\overline{S}))^\xi$ has to hold true, cf. (2.16).

Using the explicit expression for the function f introduced in Section 2.3, the equation $\dot{\lambda}_S = 0$ yields

$$\begin{aligned} 0 = &- \theta \widehat{X}_B \left(\widehat{\lambda}_H \frac{\partial g_H}{\partial S} + \widehat{\lambda}_L \frac{\partial g_L}{\partial S} \right) \\ &+ \widehat{\lambda}_S \left(r + \mu + 2\nu \left(\left[\widehat{\lambda}_S \cdot \frac{\nu}{2} \frac{1 - 2\widehat{S}}{(1 - \widehat{X}_H)^\xi} \right] \wedge 1 \right) \right). \end{aligned} \tag{2.22}$$

2.4.1.1 Monolingual stationary points

First we want to consider stationary points with $\widehat{X}_B = 0$. Obviously, if $X_B = 0$, then the linguistic composition does not change anymore, since families of type HL are impossible, while no bilinguals, which function as a kind of language transmitters, are part of the population. In the steady state all families are of types HH and LL and children of such families are raised monolingual in the respective language. Hence, both monolingual language groups reproduce themselves independent of the statuses of both languages. Thus, the state does not invest any money to support the status minority language, which would produce costs without having any positive effect, i.e. $\widehat{S} = \widehat{s}^* = 0$.

2.4.1.2 Bilingual stationary points

Now we want to consider stationary points with a bilingual sub-population, i.e. $\widehat{X}_B > 0$. Using the notation introduced above this yields that whenever $\widehat{X}_l > 0$, $l = H, L$, the stationarity implies $g_l(\widehat{X}, \widehat{S}) = 0$.

Bilingual stationary points with $X_L = 0$

The most interesting case is when monolingualism in the minority language L vanishes and only monolinguals in H and bilinguals remain. Such a state is desirable, since all society members are able to communicate with each other, while speakers of L can still preserve their cultural identity. If $X_L = 0$ we need $\dot{X}_L \leq 0$. This is equivalent to $\widehat{X}_H \geq \widehat{X}_H^\Delta(S)$.

Let

$$\underline{S} < S \leq \min\{\overline{S}, \widetilde{S}\},$$

and $X_H = X_H^*(S)$. The co-state equation $\dot{\lambda}_H = r\lambda_H - \mathcal{H}_{X_H} = 0$ is independent of λ_L, since $g_L = 0$ and $\partial g_L / \partial X_H = 0$, see A.1. Hence, we can derive $\lambda_H(S) = \lambda_H(X_H, S)$. Given this λ_H we can choose some λ_L such that $\dot{\lambda}_L = 0$. In A.1 it is also shown that $\partial g_L / \partial S = 0$.

To identify optimal steady states we have to distinguish two possibilities. First we can check if there is a steady state at $\overline{S}$. To do so, it has to be investigated if there exists a $\widehat{\lambda}_S > 2\frac{2\nu+\mu}{\nu\mu}(1 - X_H^*(\overline{S}))^\xi$ which solves

$$0 = \dot{\lambda}_S(\widehat{\lambda}_S) = \dot{\lambda}_S(\overline{S}, X_H(\overline{S}), \lambda_H(\overline{S}), \widehat{\lambda}_S).$$

The second case covers $\underline{S} < S < \overline{S}$. Here, let $\lambda_S(S)$ be defined by (2.20). In this case steady states can be found by identifying statuses S which solve

$$0 = \dot{\lambda}_S(S) = \dot{\lambda}_S(S, X_H(S), \lambda_H(S), \lambda_S(S)).$$

Depending on the parameter constellation and especially depending on k, ν and μ such a solution exists. If k is too small, then no such solution exists, that means it is not profitable to maintain the minority language L.

Lemma 2.4.1.1. *For k sufficiently large there exists at least one solution $\widehat{S}^* \in (\underline{S}, \min\{\overline{S}, \widetilde{S}\}]$ such that*

$$0 = \dot{\lambda}_S(\widehat{S}^*) = \dot{\lambda}_S(\widehat{S}^*, X_H(\widehat{S}^*), \lambda_H(\widehat{S}^*), \lambda_S),$$

where $\lambda_S = \lambda_S(\widehat{S}^*)$ if $\widehat{S}^* < \overline{S}$, and $\lambda_S > 2\frac{2\nu+\mu}{\nu\mu}(1 - X_H^*(\overline{S}))^\xi$ if $\widehat{S}^* = \overline{S}$.

For a proof see the Appendix.

Bilingual stationary points with $X_L > 0$
For an optimal steady state with $X_H, X_L, X_B > 0$ we need

$$\underline{S} \vee \widetilde{S} < \widehat{S} \leq (1 - \widetilde{S}) \wedge \overline{S}.$$

This is only possible if $\widetilde{S} < \overline{S} < 1/2$, which does not hold true for all parameter constellations, cf. Example 2.5.0.1.

For fix S we need the following for any steady state:

$$q_H(HB), q_L(LB), q_L(BB) > 0.$$

The last inequality is due to $S < 1/2$ and $\zeta < 1$. If $q_H(BB) = 0$, then $\zeta(1 - S) > 1/2$ has to hold true, else $q_H(BB) > 0$.

As before, for suitable S (here $\max\{\underline{S}, \widetilde{S}\} < S < \overline{S}$), we can find $X_H(S)$ and $X_L(S)$ such that $\dot{X}_H = \dot{X}_L = \dot{S} = 0$. For some parameter constellations there can be more than one stable solution $X_H(S)$ and $X_L(S)$ such that $\dot{X}_H = \dot{X}_L = 0$. Furthermore we get a unique $\lambda_S(S)$. The co-state equations yield a linear system in λ_H, λ_L with 3 equations and coefficients depending on S. To identify the optimal status, one has to check if this linear system has a solution for some suitable S. This also holds true at the left boundary. At the right boundary one has to check if the linear system in λ_H, λ_L and λ_S has a solution with a sufficiently large λ_S, see above.

2.5 Numerical calculations

In this section we numerically investigate the linguistic behavior of the population under the optimal policy. We show the existence of different stable and optimal steady states. Moreover, we illustrate the dependence of the selected steady state

	k	ξ	$\widehat{S}$	$\widehat{s}^*$	$\widehat{X}_H$	$\widehat{X}_L$	$k\widehat{X}_B - \widehat{s}^*$
Example 2.5.0.1	60	0	-	-	-	-	-
	75	0	0.41	0.74	0.85	0	10.3
		1	$\overline{S} \approx 0.42$	1	0.81	0	13.4
	90	0	$\overline{S} \approx 0.42$	1	0.81	0	16.3
Example 2.5.0.2	20	0	0.47	0.03	0.44	0.03	10.6

Table 3: Stable bilingual steady states for Examples 2.5.0.1 and 2.5.0.2. This table contains stable bilingual steady states – if such exist – for different values of k. The steady state values of the status $\widehat{S}$, the optimal control $\widehat{s}^*$, the fraction of speakers $\widehat{X}_H$ and $\widehat{X}_L$ as well as the steady state objective $k\widehat{X}_B - \widehat{s}^*$ are listed. Here $r = 0.5$ and $\xi = 0$.

on the initial distribution of speakers as well as on how much bilingualism is valued with respect to expenditures by the decision maker (parameter k). To analyze the evolution towards the steady states we plot exemplary trajectories.

Two examples are considered. For both of them we set $\eta = 0$. In Example 2.5.0.1 we choose μ, the rate of decline of the minority language status S, to be 0.2, which is relatively high. In contrast, Example 2.5.0.2 depicts a case where the status of the minority language declines rather slowly over time ($\mu = 0.01$). Furthermore, the parameter ζ, which measures the aggregated weight that is put on the status in the decision of lB families, $l = H, L$, to socialize their children as monolinguals in l, is slightly higher in Example 2.5.0.1. In both examples we chose the discount rate r to be 0.5.

Example 2.5.0.1. $\beta = 0.4$, $\delta = 0.7$; $\gamma = 0.1$, $\varepsilon = 0.4$, $\zeta = 0.8$; $\nu = 0.5$, $\underline{\mu = 0.2}$, $\theta = 1$ *and* $\xi = 0$

Example 2.5.0.2. $\beta = 0.4$, $\delta = 0.7$; $\gamma = 0.1$, $\varepsilon = 0.4$, $\zeta = 0.7$; $\nu = 0.5$, $\underline{\mu = 0.01}$, $\theta = 1$ *and* $\xi = 0$

First we calculate the S- thresholds, cf. Section 2.3.2. In Example 2.5.0.1 we have $\underline{S} = 0.375$, $\overline{S} = 0.417$ and $\widetilde{S} = 0.492$, while in Example 2.5.0.2, $\underline{S} = 0.286$, $\overline{S} = 0.495$ and $\widetilde{S} = 0.463$. According to these numbers and the statements made in Section 2.3.2, stable equilibria with $X_H, X_B > 0$ and $X_L = 0$ are possible for both examples. In Example 2.5.0.2 furthermore equilibria with $X_H, X_B > 0$ and $X_L > 0$ are possible, since $\widetilde{S} < \overline{S}$. This is not the case for Example 2.5.0.1, since there $\overline{S} < \widetilde{S}$. The actual stable bilingual equilibria are displayed in Table 3. For Example 2.5.0.1 we investigate the influence of different values of k, namely $k = 60$, $k = 75$ and $k = 90$. For Example 2.5.0.2 we concentrate on the case of

$k = 20$. For any parameter constellation there also is a manifold of steady states at $(\widehat{X}_H, \widehat{X}_L, \widehat{S}) = (\widehat{X}_H, 1 - \widehat{X}_H, 0)$, where $\widehat{X}_H$ can take any value between zero and one. In these steady states it is optimal to have $\widehat{s} = 0$. Note, however, that not every point on this manifold is a candidate for the optimal long run solution due to its stability properties, cf. Lemma 2.3.1.2. Next, we analyze the two examples in greater detail.

Example 2.5.0.1, $k = 60$

If k is small the decision maker does not have a particularly high incentive to support the status of the minority language in the long run. As can be seen in the first row of Table 3 there is no bilingual steady steady. The following happens. Let us consider a situation where the fraction of H speakers, X_H, is relatively high, while X_L and X_B and the status variable S are small. Because of the dominance of H speakers, most families are of type HH. Thus, X_H increases. Initially X_B decreases due to the low status of L and the low chances of H speakers of meeting a bilingual partner. This development is challenged by the decision maker who invests much into raising the status of L. Under such a policy the incentive to raise their children bilingual increases for HB and BB couples. This yields an increase in the number of bilinguals. Another effect of this is that LB couples have a stronger incentive to raise their children as L-monoglots. However, since the fraction of L speakers (monolingual and bilingual) is small, the policy does not have a strong effect on the overall development of the language and over all X_L decreases even further. As a result, it soon does not pay off anymore to invest into the status of the language as these measures affect less and less people. Thus, the status of L decreases again. Consequently, the incentive to raise children bilingual and therefore the fraction of bilinguals decreases as well. In the long-run, the majority of the population only speaks H and bilingual speakers disappear completely. This behavior is illustrated in Figures 2 and 3.

Example 2.5.0.1, $k = 75$**,** $\xi = 0$

Table 3 shows that for $k = 75$ there exists a steady state with 15% bilinguals and no monolingual speakers of the minority language L. To obtain this fraction of bilingual speakers in the long run, 75% of the budget has to be used. If this bilingual steady is reached or not depends on the initial state values. For the initial states considered in Figures 4 and 5 the system converges to that steady states. If the initial X_L, X_B and S would be even smaller than in Figure 4, the system is likely to converge to a steady state with almost only H-monolingual speakers, few L-monoglots and no bilinguals.

For the base case ($X_H = 0.85$, $X_L = 0.05$, $S = 0.1$), see Figure 4 and the left panel of Figure 6, the fraction of the bilingual population first decreases, since the status of language L is low, as are the fractions of L speakers and bilinguals, so the majority of couples consists of H speakers. Due to the dominance of HH couples and the high likelihood that HB and BB couples raise their children as

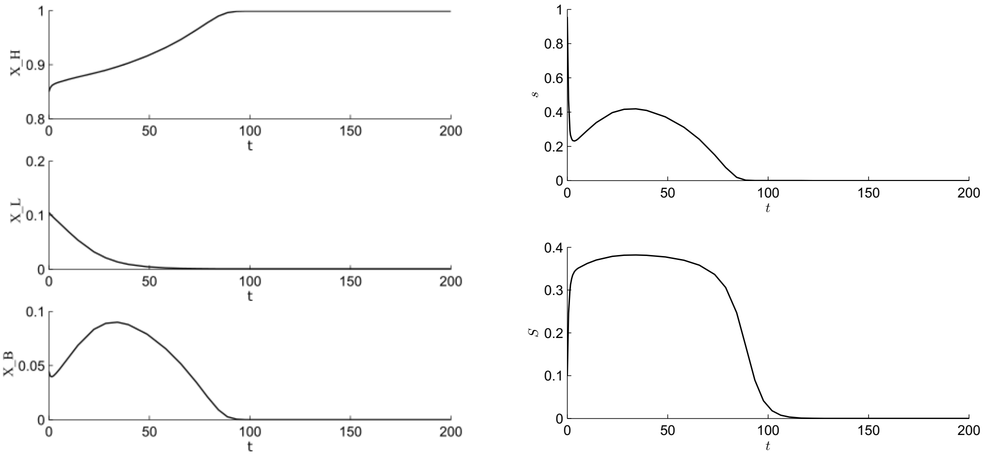


Figure 2: Time path for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.1$ (Ex. 2.5.0.1, $k = 60$).

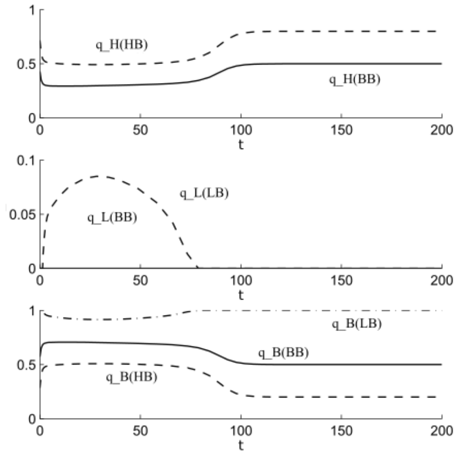


Figure 3: q-functions for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.1$ (Ex. 2.5.0.1, $k = 60$).

H-monoglots, X_H first increases. Initially one would invest as much as possible into the status to increase it. As a first result of this policy LB couples get a stronger incentive to raise their children as L-monoglots. Furthermore, HB and BB couples become less likely to raise their children just as speakers of language H and instead are more likely to raise the children bilingual than before. Consequently, X_H now decreases while X_B increases see Figures 4. Hence, the negative term in $q_L(LB)$ decreases and even more LB families raise their children as L's. This is a problem as long as X_L, which is continuously decreasing, is above some threshold. To avoid this effect, the increase of S is slowed down for a while, until X_L is small enough and then increased again to obtain the steady state status.

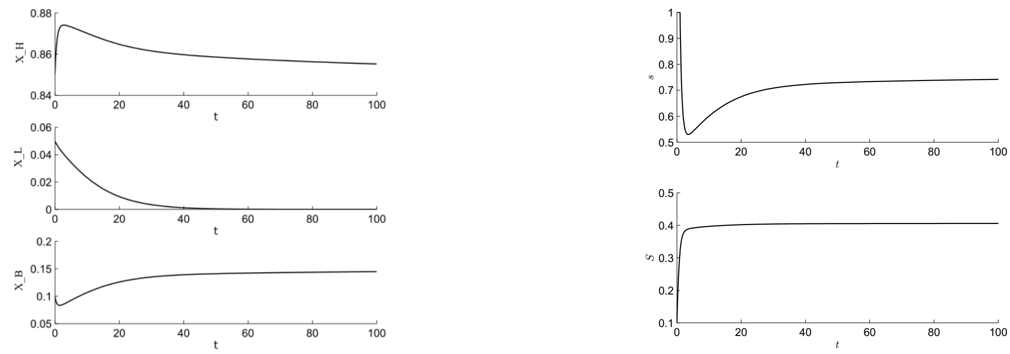


Figure 4: Time path for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.1$ (Ex. 2.5.0.1, $k = 75$, $\xi = 0$).

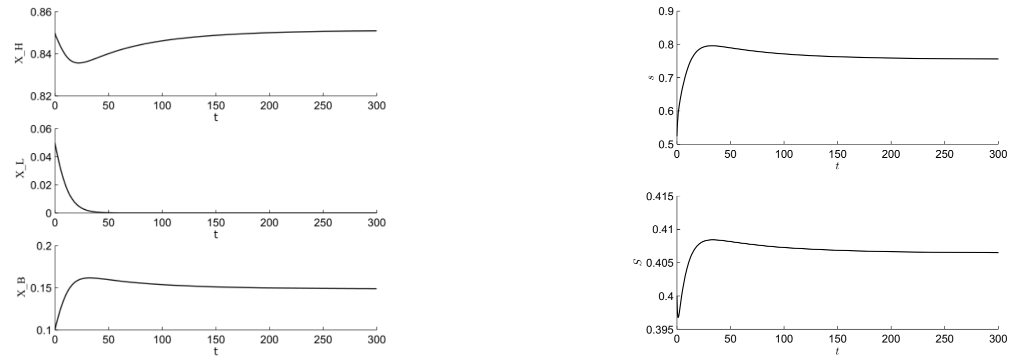


Figure 5: Time path for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.4$ (Ex. 2.5.0.1, $k = 75$, $\xi = 0$).

If, in contrast to the base case, the initial status is high, see Figure 5 and the right panel of Figure 6, then initially the state does not have to invest as much into increasing the status of the minority language. Due to the high status of L, many HB couples will raise their kids bilingual. As a result, at the beginning X_H decreases while X_B increases. Furthermore, the fraction of language L speakers is so low that LL and LB couples are rather unlikely and X_L decreases. To further support the growth of X_B it is optimal to increase s for some time. Due to the smaller fraction of L speakers, HH and HB couples are more likely than LB or BB couples, thus, X_H recovers after some time and even grows. At some point of time the status S and the fraction of bilingual speakers X_B is high enough while X_L is very low, such that s can be lowered again until it reaches its steady state.

Example 2.5.0.1, $k = 75$**,** $\xi = 1$

For $\xi = 1$ the costs for state intervention increase with the number of speakers of L, i.e. L-monoglots as well as bilinguals. Thus, the higher X_H, the lower are

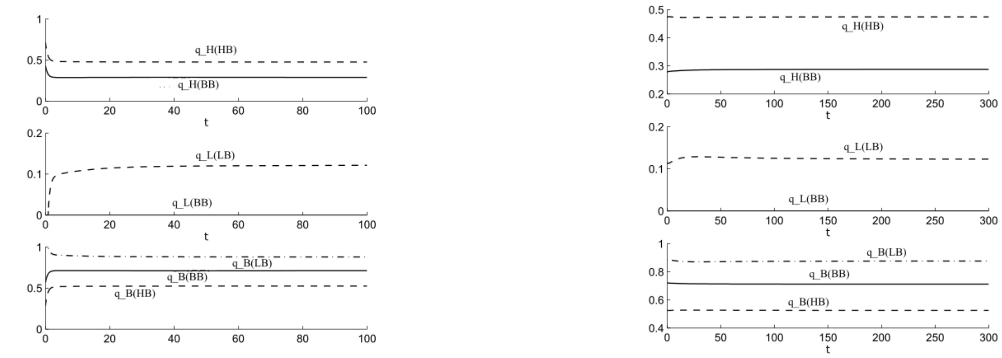


Figure 6: q-functions for Ex. 2.5.0.1, $k = 75$, $\xi = 0$).
In the both panels $X_H(0) = 0.85$ and $X_L(0) = 0.05$. In the left panel $S(0) = 0.1$, while in the right one $S(0) = 0.4$.

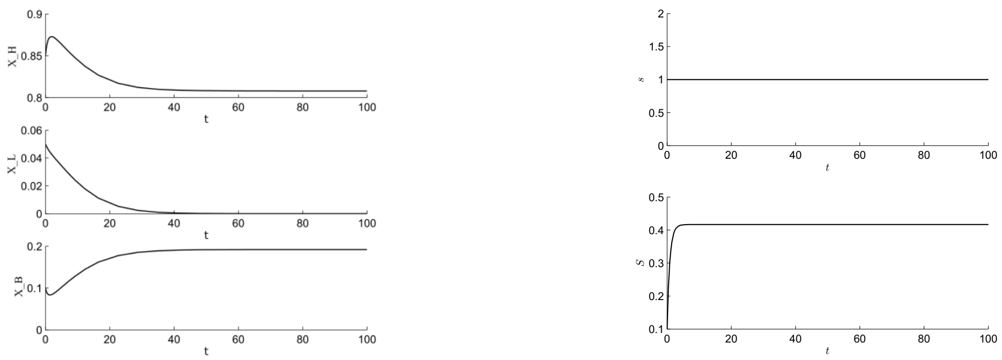


Figure 7: Time path for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.4$ (Ex. 2.5.0.1, $k = 75$, $\xi = 1$).

the costs for state intervention. In Figure 7 we can see that for the base case, the system behaves quite similar to the case of $\xi = 0$. The major difference is that state intervention is not just maximal in the beginning, but the entire budget is used over the entire time horizon. Due to the large amount of H-monolinguals the intervention is much cheaper compared to the case where $\xi = 0$ (more than 80% cheaper). Therefore, in the long run the status and X_B are higher while the X_H is smaller, cf. Table 3.

Example 2.5.0.1, $k = 90$

If k is large, then it is optimal to approach a steady state where the state invests the entire budget to reach the maximal possible status for minority language L, see Table 3. This yields a maximal amount of bilingual speakers while no L-monolinguals remain within the population. For the base case, see Figure 8, initially the state spends as much as possible for improving the status of L. For

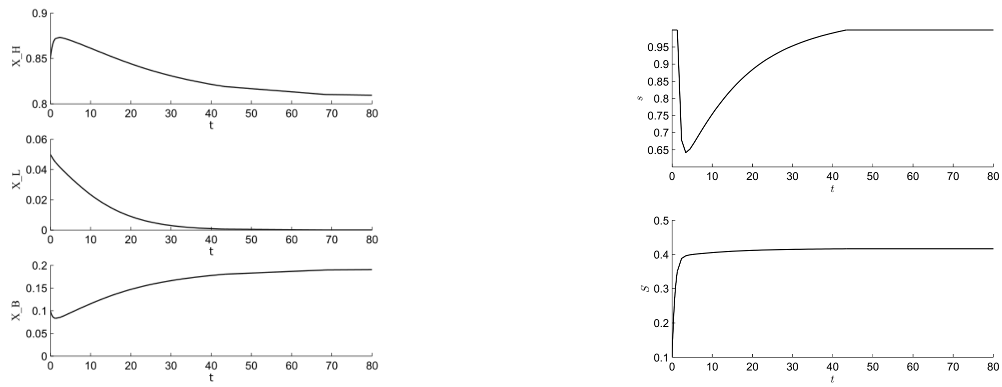


Figure 8: Time path for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.1$ (Ex. 2.5.0.1, $k = 90$).

similar reasons as before, X_H first increases while X_L and X_B first decrease. This changes after some time. Once X_L has become small enough, the state can afford to decrease efforts. However, to ensure a growth in the number of bilingual speakers, it is necessary to increase expenditures after some time again. This is the main difference to the case with a low k; where one would first decrease, then increase, and then decrease the expenditures s. I.e. the later increase is apparently necessary to reach a steady state with a proper bilingual population.

Example 2.5.0.2, $k = 20$

Table 3 shows that in the bilingual steady state for the parameter constellation considered in Example 2.5.0.2 all three linguistic repertoires remain intact in the long run. This is the major difference to Example 2.5.0.1 and is mainly due to the much lower value of μ ($\mu = 0.2$ in Example 2.5.0.1 and $\mu = 0.01$ in Example 2.5.0.2). Here with the low μ it is much less costly to keep the status at a high level. The development of the population groups is similar to before, however, X_L only decreases for a certain time, then the status of language L is so high that even BB couples have a small incentive to teach their children only language L. Due to the small depreciation of S it is not necessary to spend much for keeping the status high, so one would only invest much into the status in the beginning to get it to a high level and then decrease control efforts over time. Example 2.5.0.2 with $k = 20$ is visualized in Figure 9. Note, in the long run only 3% of the budget is used to guarantee that more than half of the population is bilingual.

2.6 Conclusions

The state aims at ensuring wide communication possibilities, while recognizing and supporting — if this is not *too costly* — minority language rights. This trade-off between a commonly spoken language and the preservation of a minority language is approached through bilingualism. To investigate how language policies

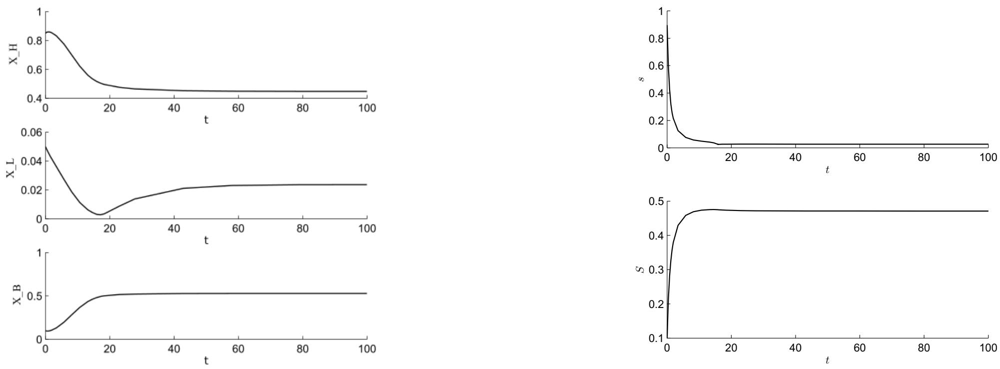


Figure 9: Time path for initial state $X_H(0) = 0.85$, $X_L(0) = 0.05$, $S(0) = 0.1$ (Ex. 2.5.0.2, $k = 20$).

can be used to preserve a minority language in a bilingual subpopulation we developed an abstract language dynamics model. The point of departure is individual utility maximization, while here only intergenerational language transmission is considered. Families decide to bring up their children either as monolinguals in the majority or the minority language, or as bilinguals. This decision is based on how they value the communicational value of each language and their emotional attachment to the languages at hand. Through a continuous investment into language policies the state can increase the status of the minority language and thereby foster bilingual parenting in families with one or two bilingual parents. It is assumed that the state wants to maximize the number of bilingual speakers at minimal costs.

In Wickström (2005) it was already proven that for a constant status and proper parameter constellations stable bilingual steady states are possible. Here we could furthermore show that such bilingual steady states can even be optimal when costs for language policies are taken into account. It was illustrated that for some cases there are steady states only with monolingual speakers of the majority language and bilinguals but without any monolingual speakers of the minority language. In such a state all individuals within the population can - in principle - communicate with each other while the minority can preserve its language. For other cases we could see that small subpopulation with monolingual speakers of the minority language survives in the long run optimal state. As one would expect, bilingual steady states are only optimal, if bilingualism is valued high enough in comparison to expenditures.

Whether or not a bilingual steady states is not only possible but really targeted by the decision maker, depends on the initial distribution of speakers as well as the initial status of the minority language. If both the status and number of speakers of the minority language are too low, then it is not worthwhile to invest in language maintenance in the long run, which results in a purely monolingual population. In most of the examples considered in the numerical analysis, the initial values were high enough and it was illustrated how expenditures change over time to achieve an optimal bilingual steady state in the long run.

For future research the current model will be extended. To get closer to the real-world complexity of language acquisition and transmission within a large population, we will add to the model language learning in formal education as well as adult language learning. Furthermore, language policies will be investigated in greater detail. We also intend to adjust the model to cases of *new* minorities, that means minorities which are based on temporary or permanent migration.

3 A Language Competition Model for New Minorities **

3.1 Introduction

Language competition or language dynamics models are formal mathematical models that describe how the size of certain language groups or the geographical distribution of certain languages change over time within a given territory. In the past two decades a growing number of such models were developed. The majority of models deal with the decline and death of languages historically rooted in specific regions. In contrast, we propose a model for minority languages predominantly spoken by migrants and their descendants, like Turkish in Germany or – to a certain extend – Spanish in the United States.

In the literature, two main strands of research can be distinguished: one influenced by economic theory, the other by physics and biology. Economic approaches are characterized by individual agency: agents aim at maximizing their individual utility when making language-related decisions. In the second strand of research, metaphors and models from physics (e.g. interacting particles) or biology (e.g. predator and prey) are adopted to describe language competition. We take a closer look at a few examples from both strands of research.

Grin (1992) presented one of the first economic language dynamics models. He considers a context with two languages. Bilingual agents can conduct various activities in one of two languages to gain utility. The efficiency of using the minority language depends on its linguistic vitality, which is determined by the number of speakers and use of the language throughout the society. The author presents a dynamic model for vitality, use and number of speakers of the minority language. This research suggests that given a minimal amount of speakers and a positive attitude to use the language, the minority language can survive (stable equilibrium). John & Yi (1996) analyzed a stylized two period model for two languages and two locations. In the first period of their model, agents can either engage in production in their location or learn a second language. Between periods, agents can move to the other location, and engage in production wherever they chosen to reside in the second period. An agent a is more productive the more producing agents there are in the same location sharing at least one language with a. The authors demonstrate that four different equilibrium outcomes are possible, including full assimilation as well as geographic and linguistic isolation. In 2001 John and Yi presented a fully dynamic version of the two-period model with successive generations. The model developed by Tamura (2001) also considers two languages and two regions. Both regions are monolingual and have growing but initially autarkic producing economies. Eventually, the optimal market size maximizing per capita income exceeds the population size of both regions, but monolingualism prevents market integration. At the individual level, parents invest in the human capital of their

** This is a manuscript of the following article: Templin, T. (2018). A language competition model for new minorities. Rationality and Society, 31(1), 40–69. Copyright ©[2018] (Sage). The final version is available at DOI: 10.1177/1043463118787487 children, caring about their own consumption as well as their children's future income. Since language skills are part of human capital and since bilinguals can function as translators and enable trade, educating children bilingually potentially increases their future income but also requires higher investments. Tamura concludes that over time an equilibrium is reached in which part of the region with the smaller population is bilingual. Kennedy & King (2005) analyze a dynamic model for a single location with three overlapping generations and a voting mechanism. As in John & Yi (1996), an adult agent is more productive when there are more other agents with whom she can communicate. Therefore, young adult agents benefit from educational policies that teach the language(s) they speak to the next generations. Kennedy and King assume that the government collects a lump-sum tax, spends the revenue on language education programs, and that adults vote for the size of the public language programs. Depending on the initial conditions, different democratic equilibria are possible. In Wickström (2005) and Templin *et al.* (2016) parents decide which language(s) to pass on to their children. In addition to their children's economic future, parents also care about transmitting the cultural values attached to their language to the next generation. Wickström (2005) shows that bilingualism can be stable, and Templin *et al.* (2016) illustrate that preserving the minority language can be optimal if bilingualism is valued high enough by the policy maker. Instead of language learning and transmission, Iriberri & Uriarte (2012) consider language use. In a context with a majority language spoken by everybody and a second language only spoken by a bilingual minority, Iriberri and Uriate analyze language choices of bilingual agents. Without any information on language(s) spoken by the interlocutor, bilinguals have to decide whether to start a conversion in the majority or in the minority language. Assuming a preference to speak the minority languace, the authors show that the bilingual population is optimally partitioned into two groups. Bilinguals in the first group hide their bilingualism and always use the majority language, while bilinguals in the second group always use the minority language.

A seminal work for the second strand of research – influenced by physics and biology – was a short paper by Abrams and Strogatz published in 2003. They proposed a simple model in which two languages compete for monolingual speakers. The attractiveness of a language is determined by the number of its speakers as well as its "perceived status". Although the authors fit their model to aggregated empirical data of endangered languages, it neglects bilinguals and predicts the extinction of one the two competing languages. Due to such shortcomings many authors, especially from the field of (statistical) physics, revised and extended the Abrams-Strogatz (AS) model. For example, Mira & Paredes (2005), Minett & Wang (2008), Heinsalu *et al.* (2014) and others extended the AS model to consider bilinguals. The role of bilinguals in language shift is crucial, since in practice individuals normally do not change from being monolingual in one language to being monolingual in the other language. Instead, it is more realistic to consider transitions from monolinguals in one language to bilinguals and - in a second step - from bilinguals to monolinguals in the other language. For such a two step transition it normally takes more than one generation. For that reason, some authors explicitly examined intergenerational language transmission, either with a uni-parental model (e.g. Minett & Wang (2008)) or with a two-parents model (e.g. Fernando *et al.* (2010)). Minett & Wang (2008) additionally consider horizontal language transmission: during their lifetime monolingual adults can learn a second language and therefore become bilingual. Other extensions of the simple AS model add a spatial dimension (e.g. Patriarca & Leppänen (2004) and Patriarca & Heinsalu (2009)) or introduce the idea of similarity between languages (Mira & Paredes (2005)). Stauffer *et al.* (2007) and others propose micro-level versions of the AS model and apply simulation techniques instead of averaging over the whole population. Pinasco & Romanelli (2006) take a somewhat different approach to the AS model. To model language dynamics, they employ a Lotka-Volterra type model[15]. Kandler & Steele (2008) and Kandler *et al.* (2010) spatially extend Pinasco and Romaneli's model. Zhang & Gong (2013) also use a Lotka-Volterra type model and offer an alternative to abstract status parameter that can be found in many AS-type models.

For more extensive literature reviews, see Patriarca *et al.* (2012), Gong *et al.* (2014) and John (2016).

In this essay we propose a model that combines aspects of both strands of research. We consider a single location with monolinguals of two languages as well as bilinguals. Adult individuals form families (two-parent model) and transmit one or both languages to their children. Furthermore, we take two different modes of horizontal language transmission into account. On the one hand, adult individuals can learn new languages to improve their human capital. On the other hand, children can acquire additional language skills in formal education. In contrast to Kennedy & King (2005), we consider the education policy as an exogenous factor. To obtain mathematical models for the different processes of language transmission and acquisition we do not use physical or biological analogies. Instead, we conceptualize families and adult individuals as utility maximizing agents. As in Wickström (2005), this utility-maximization approach helps derive general properties of the transmission and acquisition processes. Later on in the essay, we propose specific functional expressions satisfying these general properties. Moreover, we offer a possible operationalization of the most often abstract and vague status or prestige parameter used in many models that are part of the second strand of research. Central to the present model is our assumption of a steady inflow of new individuals speaking a new minority language. We assume that migration is exogenous and we analyze its effect on the language dynamics.

The essay is organized as follows. In Section 3.2 we discuss the main variables, parameters and mechanisms of the language dynamics model that is presented in Section 3.3. In Section 3.4 we suggest specific functional forms for the general model, and in Section 3.5 we discuss steady states of the model. In Section 3.6 we consider Spanish and English in the US as a numerical example to investigate

[15] A set of differential equations that are used in evolutionary biology to describe the dynamics of biological systems in which two species compete with each other, normally one as a predator and one as a prey.

the behavior of the model proposed and compare its projections to the real world developments.

3.2 Towards a Multidimensional Model

This section is devoted to the main variables, parameters and mechanisms of our language dynamics model. We start with *the number of speakers* as the central object of interest. Thereafter, we discuss how the dynamics are conceptualized and how the linguistic environment is operationalized. In conclusion, we discuss the processes of family formation, language transmission and language acquisition in greater detail.

3.2.1 Numbers of Speakers and Linguistic Composition

We are interested in the evolution of the distribution of certain language repertoires[16] throughout a population or a given territory. Like the majority of models available in the literature, we consider two languages: a locally dominant high-status majority language, H, and a low-status minority language, L. Here, high and low status is determined from a specific context, since L might be the dominant language in another country. We consider three language repertoires: monolingualism in H, monolingualism in L and bilingualism. Individuals are grouped together according to their language repertoires. By N_H we denote the number of H-monolinguals, by N_L the number of L-monolinguals and by N_B the number of bilinguals. The total population size is $N = N_H + N_L + N_B$. Throughout the essay we also use the fractions $X_H := N_H/N$, $X_L := N_L/N$ and $X_B := N_B/N$. Note, $X_B = 1 - X_H - X_L$. The vector $X = (X_H, X_L)$ determines the linguistic composition of the population.

3.2.2 Dynamics and linguistic environment

The linguistic composition of a population evolves due to a number of complex processes. Since every model builds on a necessary simplification of the complex reality, we concentrate on four key processes.

1. *Population dynamics*. This process comprises births and deaths within the population as well as migration in and out of a population
2. *Language transmission*. Parents transmit languages to their children.
3. *Education*. Pupils are educated in a certain language and can learn additional languages in formal education.

[16]Here, we use the term 'linguistic repertoire' in the very narrow sense as the set of languages a person commands, while only the two languages H and L are taken into account.

4. *Adult language learning*. Adult individuals can learn new languages, improve their skills in a language, or can forget a language they do not use regularly.

While we assume population dynamics and language learning in formal education to be exogenous, see below, we conceptualize language transmission and adult language learning as utility maximizing decisions of parents and adults. Decisions on which language(s) to transmit to the next generation and on whether to learn an additional language are made within a given linguistic environment.[17] From an individual point of view, the linguistic environment is exogenous. Nevertheless, individuals actively shape the linguistic environment through their decisions. To put it differently, individual language-related behavior is not isolated but affected - at least to some extent - by the behavior of the rest of the population. This is modeled as follows. At each point in time t individuals in a population make language-related decisions. These decisions are framed by the linguistic environment at time t. In turn, the linguistic behavior of the population, understood as aggregated individual decisions, together with population dynamics may change the linguistic environment at the next point in time $t+1$. This influence of today's behavior on the behavior of tomorrow reflects network effects.

We model five central elements of the linguistic environment. First and most obvious, the linguistic composition of the population shapes the linguistic environment. Second, we consider spatial linguistic concentration: are the speakers of the minority language distributed equally over the territory or concentrated in some areas. Third, the socio-economic as well as the official status of the two languages are important determinants of their usefulness and attractiveness and hence part of the linguistic environment. The last two elements are the language (in) education policy and language policies concerned with language learning by adults. In the following paragraphs we explain how the linguistic concentration and the status are operationalized. We discuss language education and adult language learning later on.

3.2.2.1 Linguistic concentration

We measure concentration of speakers of language L, i.e. L-monolinguals as well as bilinguals, by a single variable $C \in [0, 1]$. For $C = 0$ the language repertoire groups are distributed equally throughout the territory of consideration (no concentration). If $C = 1$, then we are dealing with a spatially segregated population (maximal concentration). We measure concentration by the index of dissimilarity, which measures the proportion of the minority population that has to move to achieve an equal distribution of L's, see Morrill (2016).

[17] Linguistic environment is a "[t]heoretical construct used for analytical purposes. It subsumes in an extensive (but obviously not exhaustive) fashion all the relevant information about the status, in the broadest sense of the word, of the various languages present in a given polity at a certain time. This includes the number of speakers, individual proficiency levels in the various languages, the domains of use of each language by different types of actors (individuals, corporations, the state, civil society organizations), and their attitudes towards the languages considered" (Grin & Vaillancourt 1997, p. 49).

3.2.2.2 Status variables

Status or prestige parameters can be found in many language competition models, e.g. in Abrams & Strogatz (2003),Wickström (2005) or Minett & Wang (2008). Most often, these parameters remain abstract and no proper interpretation of their values is provided. Therefore, Fernando *et al.* (2010) rightfully asked "what were the characteristics of a language having a prestige value, say 1.2, and what was the sociocultural condition corresponding to the difference between two languages having prestige values, say 1.2 and 1.3, respectively?" (p. 50). A similar criticism is expressed in Zhang & Gong (2013). To get a more meaningful status variable, we turn to the systematic framework first developed in Giles *et al.* (1977). There, the authors suggest three categories of factors involved in language vitality: demographic factors, status factors and institutional support factors. Demographic factors are related to the number of speakers as well as their spatial distribution, and are captured in our model by the linguistic composition and the concentration variable. The main status factors are economic, social and symbolic status. The economic status displays the economic standing of the language groups, the social status is related to prestige, social standing and (political) power and the symbolic status is related to identity and culture, cf. Baker (2011, pp. 55f). Institutional support comprises government (services) on different levels, mass media, business and education. For the present model we concentrate on the socio-economic and the institutional dimension,[18] proposing status variables for both dimensions.

The socio-economic position of a language is measured by the average socio-economic status (SES) of all it's speakers. The SES is a widely used variable in econometric and sociological analyses and can be measured in different ways. The main dimensions of the SES are income, education and occupation. Let $\overline{S}_{SE}(L)$, $\overline{S}_{SE}(H) \in [0,1]$ denote the average normalized SES of languages L and H. As before, bilinguals are counted as speakers of L. Since we are interested in the socio-economic advantage individuals gain from speaking a certain language, we consider the relative status $S_{SE}(L) := \overline{S}_{SE}(L)/(\overline{S}_{SE}(L)+\overline{S}_{SE}(H))$ and $S_{SE}(H) := 1 - S_{SE}(L)$

The *institutional* or *official* status of a language is determined by *official* domains the language can be used for.[19] We denote the number of domains taken into account by d, and make the simplifying assumption that for every domain a language can either be used or not used. Counting the number of domains a language l, $l = H, L$, can be used for and dividing this number by d, we obtain a status variables $\overline{S}_{OF}(l)$ with possible values in $\{0, 1/d, 2/d, ..., 1\}$. As for the socio-economic status, we introduce the relative official status $S_{OF}(l) = \overline{S}_{OF}(l)/(\overline{S}_{OF}(H)+\overline{S}_{OF}(L))$, $l = L, H$. Three possible domains are: government at national level, government at provincial level and public administration.

To simplify the formulas in the upcoming section we introduce a single status indicator that is composed of our two status parameters. Let α_{SE} and α_{OF} be

[18] Measuring the symbolic status of a language imposes various difficulties. "The components of non-market value [of a language] are very difficult to identify theoretically, and no less difficult to measure empirically [...]" (Grin 2003, p. 38).

[19] Comparable to the graded intergenerational disruption scale (GIDS), see Fishman (1991).

weights, i.e. $\alpha_{SE}, \alpha_{OF} \in [0,1]$ and $\alpha_{SE} + \alpha_{OF} = 1$. Given such weights we define the composed status indicators

$$S(l) := \alpha_{SE} \cdot S_{SE}(l) + \alpha_{OF} \cdot S_{OF}(l), \tag{3.23}$$

$l = H, L$. By construction, $S(l) \in [0,1]$, $l = H, L$, and $S(H) = 1 - S(L)$. The composed indicator $S(L)$ reflects the relative linguistic disadvantage of speakers of L in terms of socio-economic standing and access to public institutions and services in their frist language. Throughout the essay we simply write S for $S(L)$.

3.2.3 Family Formation and Intergenerational Language Transmission

Language transmission within the family context is one of the most important factors for language vitality. We assume that families consist of two adults and one or more children. Children are either brought up as monolinguals in H or L, or as bilinguals. The linguistic repertoires of the parents affect which of the two languages a child acquires. Therefore, we first model the formation of parent-couples before dealing with the transmission of languages.

3.2.3.1 Family Formation

Not differentiating between the language repertoire of the mother and the father, there are six possible parent-couple/family types F, where $F = HH$, HB, LL, LB, BB, and HL. Family formation is assumed to be the result of a random search and mating process: adult individuals meet randomly, some form couples and some of these couples have children. Since both parents shall be able to communicate with each other, we neglect the unlikely family type HL. We are interested in the fraction of families of type F, $F = HH, HB, LL, LB, BB$, denoted by $\psi(F)$. These fractions depend on the linguistic composition of the population X as well as on linguistic concentration C. Hence, $\psi(F) = \psi(F; C, X)$. The higher the linguistic concentration, the lower the expected number of linguistically mixed families. The random family formation process yields the following distribution of family types ψ:

$$\psi(HH; C, X) = \big(C + (1-C)X_H\big)X_H + (1-C)X_H X_L, \tag{3.24}$$

$$\psi(HB; C, X) = 2(1-C)X_H X_B, \tag{3.25}$$

$$\psi(LL; C, X) = \left(1 + C\frac{X_H}{1 - X_H}\right) X_L^2 + (1-C)X_H X_L, \tag{3.26}$$

$$\psi(LB; C, X) = 2\left(1 + C\frac{X_H}{1 - X_H}\right) X_L X_B, \tag{3.27}$$

$$\psi(BB; C, X) = \left(1 + C\frac{X_H}{1 - X_H}\right) X_B^2. \tag{3.28}$$

See Section 5.2.2 for a justification of the (3.24)-(3.28). Note, for $C = 0$, i.e. no linguistic concentration, the family type distribution is the same as given in Templin *et al.* (2016) and Chapter 2.

3.2.3.2 Intergenerational language transmission

For all the theoretically possible combinations of parents' and children's language repertoires we want to know how frequent they occur in a given linguistic environment. The frequency of F-type families who decide to bring up their children with repertoire R is denoted by $q_R(F)$. We make three simplifying but sound assumptions

A1: All children in one family have same language repertoire.

A2: Both parents shall be able to communicate with their children, i.e. $q_L(HH) \equiv q_L(HB) \equiv 0$ and $q_H(LL) \equiv q_H(LB) \equiv 0$.

A3: Only those languages spoken by the parents can be transmitted, i.e. $q_H(HH) \equiv q_L(LL) = 1$.

Since $\sum_R q_R(F) = 1$, we have $q_B(HB) = 1 - q_H(HB)$ and $q_B(LB) = 1 - q_L(LB)$.

To specify the $q_R(F)$, we adopt the model in Wickström (2005) where the author assumes that parents make a rational decision on the language(s) to raise their children in. Parents take into account the two main aspects of language: a means of communication and a carrier of identity. On the one hand, parents want to raise their children in a language that promises wide communication opportunities and few communication costs (which come into play if an L and an H monolingual individual encounter one another). We add that parents also consider the socio-economic opportunities a language offers, see the human capital argument below. On the other hand, due to identity related motives, parents are emotionally attached to their heritage language and, therefore, gain utility from transmitting it. Consequently, there might be a trade-off between instrumental and identity related motives for speakers of L. Using simple utility maximization, Wickström (2005) derives some general properties of the $q_R(F)$. Similar, but with a slightly more complex linguistic environment, we derive the following general properties, see the Appendix:

P1: The higher the number of l-monoglots, the higher the incentive to transmit l.

- P1a: If X_H increases, then q_H and q_B do not decrease and q_L does not increase.
- P1b: If X_L increases, then q_L and q_B do not decrease and q_H does not increase.

P2: The higher the status of a language, the higher the incentive to transmit it.

- P2a: If S increases, then $q_H(HB)$ and $q_H(BB)$ do not increase.
- P2b: If S increases, then $q_L(LB)$ and $q_L(BB)$ do not decrease .

P3: From properties P1 and P2 it can be deduced that $q_H(HB)$ is independent of X_H and that $q_L(LB)$ is independent of X_H.

Not every family is successful in transmitting the language repertoire they choose. Transmitting a language in a region dominated by monolinguals of the other language is especially challenging. Therefore, we introduce $Q_R(F)$, the frequency of F-type families with R-type children. Unsuccessful transmission is modeled as a question of linguistic concentration. We assume that given maximal concentration ($C = 1$), LB and BB families live in L dominated neighborhoods, and are not able to successfully transmit H to their children. Furthermore, we assume that one half of the HB families live in H-dominated areas, while the other half resides in L-dominated areas. Those in H dominated areas can not successfully transmit L. Due to these considerations we define $Q_H(HH) = q_H(HH) \equiv 1$, $Q_L(LL) = q_L(LL) \equiv 1$ and

$$Q_H(HB; C, S, X) = (1 - C) \cdot q_H(HB; S, X) + C/2, \tag{3.29}$$
$$Q_H(BB; C, S, X) = (1 - C) \cdot q_H(BB; S, X), \tag{3.30}$$
$$Q_L(LB; C, S, X) = (1 - C) \cdot q_L(LB; S, X) + C, \tag{3.31}$$
$$Q_L(BB; C, S, X) = (1 - C) \cdot q_L(BB; S, X) + C. \tag{3.32}$$

3.2.4 Language Learning in Formal Education

"One truth is certain: formal schooling introduces perhaps the most important outside influence (for better or worse) on a family's strategy to rear multi-literate/ multilingual children [...]" (Caldas 2012, p. 357). The daily language(s) of the classroom, the foreign languages taught and the language repertoires of their peers have a strong effect on pupils future language repertoires. If education is only available in the majority language, then the maintenance of the minority language is hampered. Over the years, children learn abstract ideas and concepts and their vocabulary in the majority language grows. At some point, they can express themselves better in the majority language, since the daily conversations at home might not be able to compete with this enlarged vocabulary, see, e.g., Okita (2002, p. 125). If courses or programs in the minority language are offered, the maintenance is much easier and some H-monolingual might acquire L during their school career.

For the present essay, we assume that language learning at school depends only on language education policies and the quality of language education, but not on the linguistic composition or on the status of the minority language.[20] In the model, language learning is represented by constant parameters the s_{R_1,R_2}, $R_1, R_2 = H, L, B$. Parameter s_{R_1,R_2} is the fraction of children entering school with language repertoire R_1 and leaving school with language repertoire R_2. We assume

[20]Although we model education policies as being exogenous, the linguistic composition of a democratic society can clearly affect the language dimension of these policies. This is modeled explicitly in Kennedy & King (2005). To justify constant and exogenous policies, times scales have to be adequate and the linguistic composition should not change drastically.

that neither H-monolingual nor bilingual children will unlearn H through formal education, i.e. $s_{H,L} = s_{B,L} = 0$. Hence, $s_{H,H} = 1 - s_{H,B}$ and $s_{B,B} = 1 - s_{B,H}$. Note, if $s_{H,B} > 0$, then some former H-monolingual children learn L at school. This is only possible if L is taught as a second language. If $s_{B,H} > 0$, then some children entering school as bilinguals leave the school system as H-monolinguals, which might be the case if an H-only language policy is applied. In principle we could include the option that L-monolingual children can get educated fully in L, which translates to $s_{L,L} > 0$. In practice, since H is a dominant majority language, all children learn H at some point in their educational career. Therefore, we set $s_{L,L} = 0$. Hence, $s_{L,B} = 1 - s_{L,H}$. Finally, it is reasonable to assume that bilingual children are more likely to leave school as H-monolinguals than children entering school as L-monolinguals, i.e. $s_{B,H} > s_{L,H}$.

3.2.5 Language Learning by Adults

The majority of children growing up in an environment dominated by a single language do acquire this language at some point. This is different for adults moving to such an environment (temporarily or permanently) at a certain age but with little or no proficiency in the dominant language. Even if they stay for many years, they might never acquire the local language sufficiently to be counted as bilinguals.

Learning an additional language is an investment in one's human capital. In an H-dominated environment, certain skills in H can be a precondition for participation in the local job market, and with good H proficiency one can find better jobs and/or be more productive in the job. This is especially relevant, if $S(H)$ is high. Furthermore, proficiency enables more efficient consumption and communication with locals and local authorities/administration, and reduces communication costs. This yields incentives to learn H. At the same time, language learning is costly, it requires the investment of time and other resources. We assume L-monolingual adults are rational and decide to learn H if the utility they gain (u_{LB}) is higher than the associated learning costs (c_{LB}). Utility (u_{LB}) as well as costs (c_{LB}) depend on the linguistic environment. Chiswick & Miller (2002) provide valuable insight to these dependencies. Based on a human capital approach, they analyze proficiency in H as a function of exposure to the language, efficiency in learning the language and economic incentives (three E's). Two of their many theoretical and empirically backed results can be summarized as follows. Exposure to H yields extra learning incentive and makes learning the language easier and more effective. Hence, geographic concentration of speakers of the minority language L has a negative effect on H acquisition. Moreover, the more educated people are, the more efficient they will be at learning a language. With respect to the linguistic environment variables and parameters described above we assume the dependencies of (u_{LB}) and (c_{LB}) depicted in Table 4. Furthermore, we assume the acquisition of H can be supported by language policies, e.g. by offering free language courses. We introduce a policy parameter $0 \leq v_H \leq 1$. A higher v_H indicates more support for the acquisition of H by L monolinguals (normally newcomers).

environment parameter	u_{LB}	c_{LB}
X_H	+	-
C	-	+
$S(H)$	+	(-)
v_H		-

Table 4: Changes in u_{LB} and c_{LB} for an increase in environment parameters

For the macro model, we are interested in the overall numbers of L-monolingual adults learning H, which are the aggregated result of individual decisions. We denote the annual rate of L's learning H by a_{LB}. As for language transmission, properties of a_{LB} can be derived from the utility maximization approach, see the Appendix. According to Table 4, a_{LB} decreases with an increase of C, increases with an increase of X_H, increases with an increase of $S(H)$ and increases with an increase of v_H. We also consider learning of the minority language L by the majority and construct a_{HB} analogously to a_{LB}.

3.2.6 Summary

The numbers N_H, N_L and N_B evolve due to family formation, language transmission within the family, language education and adult language learning. Therefore, four quantities determine the dynamics in the mathematical model. For the mathematical model this translates into four quantities determining the dynamics:

- ψ_F: fraction of F-type families,
- $Q_R(F)$: fraction of F-type families raising their children with repertoire R,
- s_{R_1,R_2} : fraction of children entering school with R_1 and leaving it with R_2,
- $a_{l,B}$: rate at which l-monolingual adults become bilingual.

Note, q_R, s_{R_1,R_2} and $a_{l,B}$ can also be interpreted as probabilities: $q_R(F)$ represents the probability that a child growing up in an F-type family develops language repertoire R.

3.3 General Model Formulation

Let us now combine all the pieces to built a general language dynamics model. We start with family formation and language transmission. Given the distribution of language repertoires (X) in one generation and the relevant environmental parameters (the status indicator $S = S(L)$ and the linguistic concentration C), the fraction of the next generation equipped with the language repertoire R is given by

$$\sum_F Q_R(F; C, S, X) \cdot \psi(F; C, X). \tag{3.33}$$

In the following let t be time measured in years. $N(t)$ is the overall population size at time t. $N_H(t)$, $N_L(t)$ and $N_B(t)$ denote the sizes of the language repertoire groups. The linguistic composition at time t is $X(t) = (X_H(t), X_L(t)) = (N_H(t), N_L(t))/N(t)$. With λ we denote the annual birth rate and μ denotes the annual death rate. We assume that birth and death rates are the same for all language repertoire groups. However, the model could easily be adjusted for differing death and birth rates for the different language repertoire groups and family types. Omitting mobility, language education and adult language learning for the moment, the dynamics of the basic model can be described by the following three differential equations:

$$\dot{N}_R(t) = -\mu N_R(t) \, + \, \lambda N(t) \sum_F Q_R(F; C, S, X(t)) \psi(F; C, X(t)), \tag{3.34}$$

$R = H, L, B$. The first summand represents the number of people with language repertoire R dying at time t. The second summand represents all the children raised with language repertoire R at time t. The overall population size changes according to $\dot{N}(t) = (\lambda - \mu)N(t)$. Note, to fully describe the dynamic system $\dot{N}(t)$, $\dot{N}_H(t)$ and $\dot{N}_L(t)$ are sufficient, since $N_B = N - N_H - N_L$. For better readability we introduce

$$f_R(t) := \sum_F Q_R(F; C, S, X(t)) \psi(F; C, X(t)), \qquad R = H, L, B. \tag{3.35}$$

Next, we extend the basic model step-by-step, including schooling, adult language learning and migration.

Formal education. Recall, we assumed above that $s_{H,L}, s_{B,L}, s_{L,L} = 0$. What is considered is the learning of an additional language ($L \to B$ and $H \to B$), loss of the minority language due to exclusive schooling in H ($B \to H$) as well as a switch to the majority language ($L \to H$). The basic model with education is described by

$$\dot{N}_H(t) = -\mu N_H(t) \, + \, \lambda N(t) \big((1 - s_{H,B}) f_H(t) + s_{B,H} f_B(t) + s_{L,H} f_L(t) \big), \tag{3.36}$$

$$\dot{N}_L(t) = -\mu N_L(t) \, + \, \lambda N(t) (1 - s_{L,B} - s_{L,H}) f_L(t). \tag{3.37}$$

To simplify notation even further we introduce

$$g_H(t) := (1 - s_{H,B}) f_H(t) + s_{B,H} f_B(t) + s_{L,H} f_L(t), \tag{3.38}$$

$$g_L(t) := (1 - s_{L,B} - s_{L,H}) f_L(t). \tag{3.39}$$

Adult language learning. We only consider acquisition of an additional language, i.e. the transition from monolingualism to bilingualsm ($a_{H,B}$ and $a_{L,B}$). Using the notation introduced in (3.38)-(3.39) and writing $a_{L,B}(t)$ for $a_{R,B}(C, S, X(t))$, we obtain the basic model with schooling and adult language learning:

$$\dot{N}_R(t) = -\big[\mu + (1-\mu)a_{R,B}(t)\big] N_R(t) \; + \; \lambda N(t) g_R(t). \tag{3.40}$$

Mobility. In most of the language competition models reviewed above the population is constant in the sense that no new individuals enter the population (except the ones born within the population) and that individuals do not leave the territory. However, we explicitly model an external inflow of new individuals. We focus on the migration of people with a heritage language that differs from the dominant language of the host country H. The absolute number of people equipped with language repertoire R migrating to the population at time t is denoted by $M_R(t)$. The total number of migrants at time t is given by $M(t) := M_H(t) + M_L(t) + M_B(t)$. Note, in principle $M_R(t)$ could be negative, which would indicate net emigration of R's. In the general case we get $\dot{N}(t) = (\lambda - \mu)N(t) + M(t)$ and hence, $R = H, L$,

$$\dot{N}_R(t) = -\big[\mu + (1-\mu)a_{R,B}(t)\big] N_R(t) \; + \; \lambda N(t) g_R(t) \; + \; M_R(t). \tag{3.41}$$

Additionally, we define $m_R(t) = M_R(t)/M(t)$, the share of R mobility. In the following paragraphs, we consider two special cases for migration. In the first case, we assume migration to be constant over time, i.e. every year the same absolute number of migrants enter the population (e.g. 100,000 migrants per year). In the second case, migration is constant relative to the population size. If e.g. the government allows for an annual migration of 2% of the total population size, then migration is constant relative to the population size.

Special case 1 (Constant absolute migration flow). $M_R(t) = M_R$ is constant.

Special case 2 (Constant relative migration flow). $M(t)/N(t) =: \nu$ is constant as well as the fractions $M_R(t)/N(t) =: \nu_R$.

3.4 Specific Functional Forms

To perform numerical analysis and simulations of the model, we consider some functional expressions for $q_R(F; S, X)$ and $a_{R,B}(C, S, X)$. For intergenerational language transmission, we use the functional expressions proposed in Templin *et al.* (2016). For non-negative parameters $0 < \beta < \delta$ and $\varepsilon + \gamma < \zeta < 1$ consider the functions

$$\begin{aligned}
q_H(HB; S, X) &:= \max\{0, \zeta(1-S) - \beta S X_L\}, \\
q_H(BB; S, X) &:= \max\{0, \varepsilon(1-S) + \gamma(1-S)X_H - \delta S X_L\}, \\
q_L(LB; S, X) &:= \max\{0, \zeta S - \beta(1-S)X_H\}, \\
q_L(BB; S, X) &:= \max\{0, \varepsilon S + \gamma S X_L - \delta(1-S)X_H\}.
\end{aligned}$$

These functions are constructed in such a way that they satisfy properties P1-P3 and if X_H is sufficiently high and S is sufficiently low, then bilingual (BB) and even mixed couples (LB) do not raise their children as L-monoglots. The parameters $\beta, \gamma, \delta, \varepsilon$ and ζ regulate the effects of the relative status of both languages and the sizes of the language repertoire groups on parents' decisions. If ε and ζ increase, then the effect of the status of a language for it's transmission increases. In contrast, if β and δ increase, then negative effect of the status of H, resp. L, on the transition of L, resp. H, becomes stronger.

Next, we provide specifications for the functions $a_{l,B}(S, C; X)$, $l = H, L$, that satisfy the conditions outlined above. Let θ and ϕ be parameters between 0 and 1. Support for the acquisition of H and L for monolingual speakers of the other language is modeled by the parameters v_H and v_L. We set

$$\widetilde{a}_{L,B} := \max\left\{0, \theta(1-S)X_H - \phi(1-v_H)\right\},$$
$$\widetilde{a}_{H,B} := \max\left\{0, \theta S X_L - \phi(1-v_L)\right\}.$$

For maximal support for the acquisition of the local language H for L-monoglots, i.e. $\nu_H = 1$, the function $\widetilde{a}_{L,B}$ is strictly decreasing in S and X_H. This reflects that the higher the socio-economic and communicational incentives associated with H as well as the exposure to H, the more L-monolinguals learn H. If no support is made available, then $S(H) = 1 - S$ and X_H have to be high enough such that L-monoglots acquire H. Since linguistic concentration hinders learning of the other language, we set $a_{l,B} = (1-C)\widetilde{a}_{l,B}$, $l = H, L$. Hence, given maximal concentration/segregation, no monolingual individuals learn the other language.

3.5 Steady States

In this section we investigate steady states of the linguistic composition X. Steady states are denoted by $\widehat{X}_H, \widehat{X}_L$ and $\widehat{X}_B$. We only consider the special cases with constant absolute and constant relative migration flow and distinguish two cases concerning the birth and death rate: $\lambda \geq \mu$ and $\lambda < \mu$. We assume that H remains the dominant language, i.e. $S = S(L) < 1/2$ and $s_{L,B} > 0$, and that some migrants do not speak H sufficiently to count as bilinguals, i.e. $M_L > 0$. Therefore, we can exclude steady states with only L-monolinguals ($\widehat{X}_L = 1$) and steady states with a fully linguistically segregated population ($\widehat{X}_H, \widehat{X}_L > 0$ and $\widehat{X}_B = 0$). Consequently, there are only two relevant types of steady states left: monolingual ones with only speakers of H ($\widehat{X}_H = 1$) and steady states with bilinguals.

<u>**Case 1a:** $\lambda \geq \mu$ with $M \equiv const.$</u> If $\lambda \geq \mu$, then the population size N with $\dot{N} = (\lambda - \mu)N + M$ is always increasing. Over time, the population size N becomes very large compared to M. Hence, over time, M is negligible and so are the incoming monolingual speakers of the minority language L. Therefore, the fraction of monolingual speakers of L will tend to zero ($\widehat{X}_L = 0$). Depending on linguistic concentration C and the model parameters governing language transmission, there can be a stable monolingual equilibrium ($\widehat{X}_H = 1$) or bilingual steady states ($\widehat{X}_H, \widehat{X}_B > 0$).

Case 1b: $\lambda \geq \mu$ with $M \equiv \nu N$. The story is different for constant relative migration. In that case, the number of migrants increases with the population size. We have $\dot{N} = (\lambda - \mu + \nu)N$. Therefore, every year a relevant number of speakers of L enter the population, implying that $\widehat{X}_L > 0$. Depending on the parameter constellation, two types of steady states are possible. The most likely one is a steady state with monolinguals of both languages and bilinguals. But if concentration is very high and if the identity aspect is valued much more than the instrumental value of a language, then it is possible L-speaking parents always transmit this language to their children. As a consequence, the L-monolingual and the bilingual population increases faster than the H-monolingual population and we have a steady state with $\widehat{X}_L, \widehat{X}_B > 0$ and $\widehat{X}_H = 0$.

Case 2a: $\lambda < \mu$ with $M \equiv const.$ In this case the total population size N converges to a steady state $\widehat{N} = M/(\mu - \lambda)$. As for the linguistic composition, we have $\widehat{X}_L > 0$, since every year a relevant number of speakers of L enter the population. As in case 1b, there are two possible types of steady states, depending on the parameter constellation. If speakers of L always transmit language L to their children, then not only will the fraction of monolingual H-speakers tend to zero, but the monolingual H-speaking population will die out over time, due to $\lambda < \mu$.

Case 2b: $\lambda < \mu$ with $M \equiv \nu N$. For case 2b we have to distinguish between three sub-cases. First, if $\nu < \mu - \lambda$, then the population size is always decreasing and the population will die out in the long run. For $\nu = \mu - \lambda$ the population size is constant, since migration compensates for the low birth rate. Steady states for the linguistic composition are as in case 1b. For $\nu > \mu - \lambda$ the population size increases, and steady states occur again as in case 1b.

3.6 Numerical Example

For our empirical example, we consider Spanish (L) and English (H) in the United States. Being spoken by about 35 million people, Spanish is by far the largest minority language in the US. It owes its vitality and spread to a large extend to the continuous influx of new people from Spanish speaking countries, cf. Carreira (2013). In this section, we do not provide an in-depth socio-linguistic analysis of Spanish vitality in the US, but use rough parameter estimates to illustrate and test the application of the model presented above.

Every ten years, starting in 1890, the US Census collects information on mother tongue, language skills and use. These data are not directly comparable, since questions asked and wording changed over time. From 1980 onward, the Census asked the same three questions regarding language every ten years. First, the Census asks whether a language other than English is spoken at home. Following a positive answer to the first question, the Census inquires about what language besides English is spoken at home and how well the person speaks English (‘very well’, ‘well’, ‘not well’ or ‘not at all’). Since we limit our model to two languages, we only take into account those people with English as the sole home language and those who (also) speak Spanish at home. We count a Spanish-speaking person as bilingual, if the person indicates to speak English 'very well'. All other responses are categorized as Spanish monolinguals. Unfortunately, there is no information on the Spanish language skills of those who only spoke English at home. Therefore, we assume that these people do not speak Spanish 'very well' and count them as English monolinguals. Moreover, we do not differentiate between different groups of Spanish-speakers, e.g. by country of origin (of ancestors). These definitions of English monolinguals, Spanish monolinguals and English-Spanish bilinguals yield the numbers in Table 5.

year	1980	1990	2000	2010
N	210.247	230.446	262.375	285.797
N_H	187.187	198.601	215.424	228.700
N_L	5.372	8.309	13.700	16.209
N_B	5.724	9.037	14.300	19.259

Table 5: Estimated linguistic compositon in the US (1980-2010). Estimated total numbers of English monolinguals (N_H), Spanish monolinguals (N_L) and bilinguals (N_B) in million. Sources: U.S. Census (1990), Shin & Bruno (2003), Ortman & Shin (2011).

Using detailed information on the number of Spanish-speakers at the county level (Census 2010), we estimate linguistic concentration to be $C = 0.27$. From 1980 to 2010, the birth rate declined from almost 16 births per 1000 people to 13 births per 1000 people. The model could be adapted to account for shrinking birth rates (use $\lambda(t)$ instead of a constant λ). However, we use a static birth rate from 1995, the middle of our 30 year time interval. We set $\lambda = 0.015$. Despite some fluctuations, the death rate was more stable over the years. We set $\mu = 0.0087$. Migration numbers also fluctuated over time. In the 1980s, about 5.6 million Hispanics migrated to the US, in the 1990s about 8.1 million and in the 2000s about 6.5 million.[21] For the model, we take the average of these three figures and assume constant annual migration numbers of $M = 0.674$ million people. Only a minority of Hispanic migrants speak English "very well" when migrating to the US, see e.g. Espenshade & Fu (1997). We set $m_B = 0.07$ (7% of Hispanic migrants are bilingual).

Minority languages in the US are often lost within three generations. The first generation learns some English but speaks the minority language at home, the second generation is bilingual but tends to prefer English and the third generation has little if any competences in the minority language. The dominance of English

[21] Cf. http://www.pewhispanic.org/2017/09/18/facts-on-u-s-latinos/#hispanic-pop
It is important to note that since 2007 migration especially from Mexico has slowed down significantly.

in education and the economy are two major reasons for this dynamic. Studies also show that the shift to English occurs at a slower pace among Hispanics than among other migrant groups, and that Spanish retention is strongest among Mexican Americans, see e.g. Tran (2010). Estimating the extend of language loss that is due to schooling and growing up in an English dominated environment – parameters $s_{L,B}$, $s_{L,H}$ and $s_{B,H}$ in the model– is difficult. For precise estimates, detailed quantitative data, like the Children of Immigrants Longitudinal Study (CILS), c.f. Portes & Rumbaut (2012), have to be analyzed. This is beyond the scope of this essay. Instead, we use a few general figures drawn from the aforementioned study. For 96% of children of Mexican descent (second generation), Spanish was the main language spoken at home, see Rumbaut *et al.* (2002). For other Latin Americans this figure is 85%. As young adults, about 63% of Mexican Americans in the study reported to speak Spanish very well, and only 44% of other Latin Americans. Weighting these numbers by the sizes of the respective populations, we set $s_{L,B} = 0.61$ and $s_{L,H} = 0.38$. To estimate $s_{B,H}$, we use data on third generation Hispanics. According to Center (2002), only 22% of third and higher generation Hispanics report to be bilingual. With 54% of second generation Hispanics being Spanish dominant or bilingual, we set $s_{B,H} = 1 - 0.22/0.54 = 0.59$. Last, we turn to $s_{H,B}$, that is acquisition of Spanish by monolingual speakers of English. Spanish is the most popular second language taught in US schools. According to Furman *et al.* (2007), in 2002 about 4.5% of all students in higher education were enrolled in Spanish language courses. Considering a 5:1 ratio of introductory to advanced classes, we set $s_{H,B} = 0.045/5 = 0.009$.

Finally, we estimate the relative status of Spanish. Despite not having a *de jure* official national language, English is the *de facto* official language in US government and administration. In some states, governments provide information and public services in Spanish as well, e.g. in New Mexico, Texas and California.[22] Hence, in the US as a whole, the official status of L is very low. We set $S_{OF}(L) = 0.1$. Moreover, monolingual speakers of Spanish (mostly migrants) but also bilinguals earn, on average, less than their English monolingual counterparts and migrants have, on average, a lower educational level, see e.g. Center (2002). We estimate $S_{SE} = 0.4$ and, weighting both dimensions equally, obtain an estimate $S(L) = 0.25$. For intergenerational language transmission, i.e. for the functions $Q_{LR}(F)$, we use the parameters $\zeta = 0.2$, $\varepsilon = 0.1$, $\beta = 0.8$, $\delta = 0.9$, and $\gamma = 0.05$. In contrast to all other model parameters, these five parameters are not estimated from empirical data. Their values are chosen in such a way that empirical data are matched well by the projections. A change in these parameters changes the projections produced by the model. For the present case, though, such changes are minor. If we consider a significantly different set of parameters, say $\zeta = 0.9$, $\varepsilon = 0.4$, $\beta = 0.2$, $\delta = 0.5$, and $\gamma = 0.3$, then our projections are not as accurate as for the original set of parameters. But the difference between both projections is so small that it could barely be seen in plots like the ones presented below.

[22] For the empirical analysis we neglect the special case of Puerto Rico, where Spanish is an official language spoken by the majority of the population.

Based on these rough parameter estimates, we ran the model for 1980 to 2030. The model projections are displayed in Figures 10 and 11. The projection in Figure 10 captures the observed population dynamics quite well, despite our use of static birth, death and migration rates. Figure 11 depicts the evolution of the linguistic composition. The projections produced by the model are close to the empirically observed data, while slightly underestimating the number of bilinguals within the population. If the current trend continues, then by 2030 around 84.8% of the US population with English as the sole home language or with English and Spanish as home languages is English monolingual, 6.5% is Spanish monolingual and 8.7% is bilingual. We used rough estimates from different data sources and made some simplifying assumptions to generate these numbers. For a more sophisticated application of the model, more accurate data is needed. Moreover, the model should be adapted to account for trends in birth and death rate developments as well as migration patterns and illegal migrants.

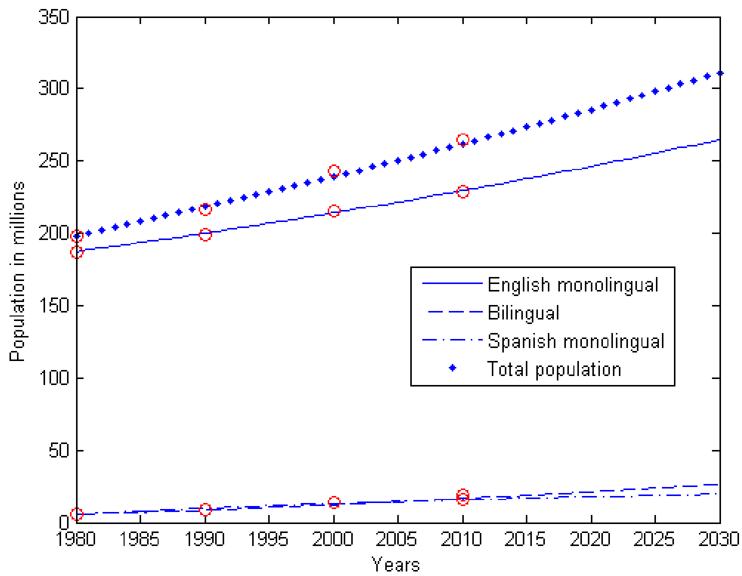


Figure 10: Projection of the linguistic composition of the US (1980-2030)- Absolute.

Numbers of English monolinguals (N_H), Spanish monolinguals (N_L) and bilinguals (N_B) from 1980 to 2030. The lines represent the numbers derived by the model; the red circles represent the empirical data listed in Table 5.

3.7 Conclusion and Outlook

In this essay we developed and analyzed a language dynamics model for new minorities that includes intergenerational language transmission, formal language education and adult language learning. Speakers of the newcomer language L enter a society with one (main) official language H, which is spoken by most society members. To improve their socio-economic position within the host country,

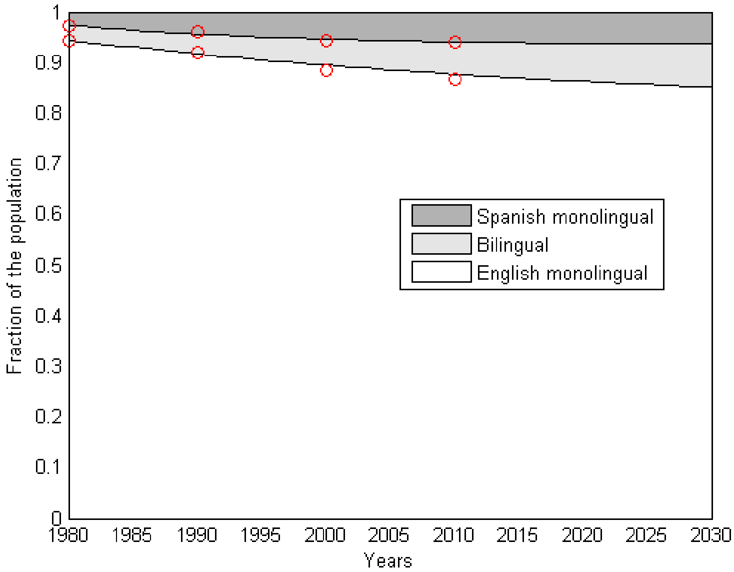


Figure 11: Projection of the linguistic composition of the US (1980-2030) - Relative.

Fractions of English monolinguals (N_H), Spanish monolinguals (N_L) and bilinguals (N_B) from 1980 to 2030. The lines represent the numbers derived by the model; the red circles represent the empirical data listed in Table 5.

some of the newcomers learn H. The model includes a parameter measuring linguistic concentration. The more segregated both language groups are, the lower the exposure to the other language and thus fewer people become bilingual. Intergenerational language transmission and language education for the descendants of newcomers prove more important for longer-term language dynamics than adult language learning. Individuals form families and transmit one or all of their languages to their children. Instrumental and emotional motives, and economic costs drive the decisions for language transmission. While the locally dominant language usually promises better socio-economic opportunities for their children, many newcomers want to transmit their heritage language to the next generation. Especially if children live in a social environment dominated by H, then transmitting the minority language might not be successful and after two or three generations the language is lost – a well known empirical observation in socio-linguistic studies. Therefore, linguistic concentration supports the transmission of L, but might hinder a proper acquisition of H.

Like Fernando *et al.* (2010) and others, we argued that a purely abstract status parameter limits the empirical applicability of language dynamics models. At the same time, we emphasized the role of factors related to the socio-economic and the institutional status for individual language related decisions. Instead of omitting the status from the model, we suggested a measurable status parameter composed of the socio-economic status of the language repertoire groups and the institutional status, which is operationalized as the number of official domains the languages can be used for. This yields an indicator of (1) the socio-economic opportunities related to a language, (2) the usefulness of a language in communication with state authorities and (3) the socio-economic resources available to members of a language repertoire group. Furthermore, the model accounts for population dynamics. The models accuracy is improved through the use of birth, death and migration rates obtained from empirical surveys.

We applied the model to Spanish and English language dynamics in the United States. We set model parameters according to rough estimates from empirical data. Census data on home language use and language skills from 1980 to 2010 inform our estimates on speaker numbers. Although a number of simplifying assumptions were made, the model could reproduce the empirical data quite well.

We see this essay as a point of departure for future research on language dynamics in societies with new linguistic minorities. To get closer to the complexity of the processes at hand, the model can be extended and refined in several ways. In the current version of the model we only consider one language minority, while normally one can observe a variety of minority language groups. In the case of multiple linguistic minorities the locally dominant language H is not only a mean of upward social mobility, but can also function as a vehicular language used in communication between speakers of different minority languages. Moreover, different levels of linguistic competences or actual language use (instead of proficiency) or multiple age groups could be modeled.

4 A Framework For Dynamic Language Policy Evaluation

4.1 Introduction

In multilingual states or regions, public authorities are required to address and manage the existing linguistic diversity. To some extent, language policies are unavoidable, even in seemingly monolingual contexts: public authorities have to communicate with the public in certain languages, at least one language has to be used in public administration and only a limited number of languages can be used on street signs and bank notes. Other important domains in which language policies play a crucial role are public education and migration. Language policies can determine the language(s) of instruction in (public) schools as well as foreign languages offered to students. Acquisition of the official/state language(s) is often an essential part of integration programs and in many countries a certain level of proficiency in the official/state language(s) is a prerequisite for the naturalization of immigrants. The latter two examples in particular show that language policies are not just sometimes unavoidable, but that they can have significant consequences for the people subject to the policies and for society as a whole.[23] Moreover, language policies normally come with certain costs. For these reasons, language policies should be selected and designed carefully. Their effects on different socio-linguistic groups as well as their implementation costs have to be considered. This requires evaluation: "Just like any other policy, language policies can (and should) be evaluated" (Gazzola 2014, p. 1).

Before we take a closer look at language policy evaluation techniques, we should clarify what we mean by the term *language policy*. In the literature, a variety of definitions can be found. A lucid overview of different approaches to define *language policy* is provided in Cassels Johnson (2013, chapter 1). Some authors work with a narrow and top-down definition of the term, for example Kaplan and Baldauf. They "portray language policy as a set of laws and regulations or rules enacted by an authoritative body (like a government) as part of a language plan" (Cassels Johnson 2013, p. 4). Other authors, especially those with a background in (socio-)linguistics or sociology, argue for a wider conception of the term or look at language policies from a different and often critical perspective. Bernhard Spolsky, for example, also includes language practices of the speech communities as well as language beliefs and ideologies in his definition. In this case, bottom-up initiatives by grassroots movements are also seen as language policies. For the present essay, we adopt a definition provided in Grin (1999) that is somewhat close to the one used by Kaplan and Baldauf. There, *language policy* is defined as "a systematic, rational, theory-based effort at the societal level to modify the linguistic environment with a view to increasing aggregate welfare. It is typically conducted by official bodies or their surrogates and aimed at part or all of the

[23]For example, "language policy in education affects the supply of linguistic skills available in a given labor market, and it can influence workers' transitional mobility decisions [...]" (Gazzola & Wickström 2016, p. 3)

population living under their jurisdiction" (18). For the current essay, we mainly think of language policies that aim at improving the status of minority languages, e.g. by granting their speakers access to public services in their first language, or at minority language maintenance, e.g. by acquisition planning.

Language policy evaluation

As with any public policy, language policies come with potential advantages and drawbacks. Consider, for example, a policy that allows citizens to access public services in both the majority and a minority language instead of just in the majority language. An obvious advantage would be that native speakers of the minority language can access public services in their first language. Typical drawbacks are the costs associated with the policy: documents now have to be produced in two languages and members of staff have to be trained to offer services in the minority language as well. Another drawback might be that a second minority language group feels even more disadvantaged after the introduction of the policy. To decide whether or not to implement a policy, or to choose between different policy options, potential advantages and drawbacks have to be assessed. To this end, both have to be quantified. Economics can be particularly helpful here, since it offers several tools and concepts to quantify, evaluate and compare alternatives. For that reason, language policy evaluation has become an important field within language economics. Despite the cultural nature of language, "the closest relative of language economics is not cultural economics, but environmental economics" (Grin 2003, p. 29). One link to environmental economics stems from the fact that advantages, and hence welfare, include non-material elements such as identity-related language issues. Therefore, concepts and methods used in (economic) language policy analysis are largely inspired by environmental economics and, more generally, by classical policy analysis. Two policy assessment methods used in language policy analysis are Cost-Benefit Analysis (CBA) and Cost-Effectiveness Analysis (CEA). In CBA benefits are expressed in monetary terms and compared to costs. In CEA benefits are not monetized but quantified in an effectiveness measure, and the quotient of this measure and monetary costs is considered. Both methods can take into account present as well as potential future costs and benefits. Accounting for the latter entails estimating future costs and benefits. Considering language policies, this task is especially delicate, since the number of their beneficiaries often changes as a result of current trends and of the policies themselves. So if costs and benefits of a policy depend on the number of beneficiaries, then they can change dramatically over time. Therefore, we argue that, for a more realistic evaluation of language policies, language dynamics have to be taken into account. This can be achieved by combining traditional and well established policy analysis tools such as CBA or CEA with language dynamics models, which are discussed in the following subsection.

Language competition models

Language competition models are mathematical models designed to analyze the evolution of interacting languages. In this essay, our focus is models that are concerned with the evolution of the number and the distribution of speakers of two or more languages within a given territory, and hence with the number of potential beneficiaries of a language policy. Languages compete for speakers in the sense that one language often gains speakers at the expense of another. In the literature, we find a variety of models. Most of them are inspired by observations and models from biology, (statistical) physics and economics. This reflects the observation that language competition shares several features with the interaction of biological species[24] or with complex systems of interacting particles.[25] Regarding economics, language competition can be theorized as the aggregated result of individual language-related decisions of utility maximizing agents within a given linguistic environment. Existing models are deterministic or probabilistic; macroscopic and based on differential equations or microscopic and based on computer simulations; consider only monolinguals of two languages or consider multiple languages and bilinguals; neglect the geographical distribution of speakers or explicitly describe the geographical diffusion of languages. The existing models yield insights into the dynamics of language evolution, change and decline. Moreover, several models, if well calibrated, can simulate and reproduce observed empirical data.

We outline some relevant models in greater detail in section 4.2. For overviews on various modeling approaches see Castellano *et al.* (2009), Gong *et al.* (2014) and John (2016).

Combining language competition models and policy evaluation

Right from the beginning, the development of language competition models was accompanied by the hope that these models "may be useful in the design and evaluation of language-preservation programs" (Abrams & Strogatz 2003, p. 900). Some scholars even explicitly model state intervention that aim to promote minority language maintenance, see e.g. Minett & Wang (2008) and Templin *et al.* (2016). Such interventions normally boil down to increasing the prestige or status – a model parameter, i.e. a single number – of the minority language, either directly, as in Minett & Wang (2008), or through unspecific investments in status planning, as in Templin *et al.* (2016). Therefore, policy recommendations and results derived from language competition models often remain quite general and

[24]"[L]anguage competition resembles the competing relation in ecology, where the rise or decline of the population size of a species is influenced by the growth rate of the competing species. This competing relation exists not only between predators and preys, but is common among various species in the biological world" (Zhang & Gong 2013, p. 9699).

[25]"In social phenomena, the basic constituents are not particles but humans, and every individual interacts with a limited number of peers, usually negligible compared to the total number of people in the system" (Castellano *et al.* 2009, p. 592).

abstract. One typical result that can be found in different versions is the following: if the status of the minority language is above a certain threshold, then the minority language can survive. Even if a numerical value for this threshold is provided, the question remains what this value actually means in practice. To be used for policy evaluation that goes beyond general and abstract statements, language competition models have to be built on parameters with a clear socio-linguistic meaning. Moreover, the parameters should be measurable in the field. This was already pointed out by Fernando *et al.* (2010) and is discussed section 4.2.

In the present essay we propose a simple macroscopic model that builds on five pivotal factors influencing the language dynamics and that only uses parameters obtainable from empirical data. The five factors taken into account here are family formation, intergenerational language transmission, language education, adult language learning and migration. They can also be seen as five central elements of the linguistic environment. The model describes how changes in such factors, i.e. changes in the linguistic environment, affect the linguistic composition of the society, i.e. the number of speakers of the different languages, over time. Since changes in the factors can be brought about by language policies, the model allows us to investigate the future effects of policies on linguistic composition. This enables us not just to evaluate the current costs and benefits of a certain policy measure, but also its future costs and benefits. Taking into account current as well as future costs and benefits is the foundation of what we call dynamic cost-benefit analysis. In a dynamic cost-benefit analysis, a policy yields a potential efficiency gain if the sum of today's benefits and (discounted) future benefits exceed the sum of today's costs and (discounted) future costs. Such a dynamic cost-benefit approach was already taken in Templin *et al.* (2016), but the policy (investing money in status planning) remained as abstract as the status parameter itself. The novelty here is that policies can be evaluated in a more realistic fashion.

The rest of the essay is organized as follows. In section 4.2, the language competition model is motivated and described. Section 4.3 illustrates the application of the model to a real case scenario. For this purpose we consider the case of Spanish and Basque in the Basque Autonomous Community in Spain. In section 4.4, we outline how the language competition model can be used to obtain a dynamic policy evaluation framework. In section 4.5, we provide an outlook on future research, with a particular focus on useful extensions of the basic language competition model, and draw some conclusions.

4.2 The language competition model

In this section, the language competition model is outlined. The model presented here stands in line with a language competition modeling tradition that started with Abrams & Strogatz (2003) and Wickström (2005). In Abrams & Strogatz (2003), two languages compete for monolingual speakers. The higher the number of speakers of a language and the higher its status, the more people it attracts. The evolution of the number of speakers is then described by a differential equation. In Wickström (2005), the language dynamics are mainly driven by family formation and intergenerational language transmission. First, adult individuals form couples. Considering monolinguals of both languages as well as bilinguals, six (linguistic) types of couples are possible. The distribution of couple types is derived from the distribution of language repertoires within the society. Second, parents decide which of their languages to transmit to their children. This decision is based on two rationales: they gain utility from transmitting languages with a wide communicational range as well as from the transmission of languages to which they are emotionally attached. In case of speakers of a small minority language, there is a trade-off between the rather small communicational value of that language and the emotional attachment to it. Moreover, it is theorized that the higher the status of the language, the higher the utility parents get from transmitting it. Conceptualized as utility-maximizing actors, parents choose the language repertoire that yields the highest utility. In the literature, extensions of both of those models can be found. In Minett & Wang (2008), for example, the authors present an extended version of the Abrams/Strogatz model that also includes bilingual speakers. Moreover, their model features horizontal language transmission. During their lifetime, monolingual adults can learn a second language and therefore become bilingual. As with the status variable, the incentive to learn a second language increases with the number of speakers of that language. An extension of the Wickström model is provided in Templin *et al.* (2016). There, the status of the minority language is a model variable that decreases over time. This decrease can be counteracted by investments in status planning measures by the state. For the current essay, we extend the model proposed in Wickström (2005). As in Minett & Wang (2008), we take into account bilingual speakers and horizontal language transmission. Moreover, we add language education in schools and migration.

As noted in the introduction, the majority of language competition models are inspired by models from physics, biology and economics. Most models feature one or more numerical parameters that often lack an explicit socio-linguistic meaning. Such parameters can have a considerable effect on the outcome of the model and are mostly obtained by fitting the model to existing data on the evolution of the linguistic composition of a given society. If language competition models are supposed to be used for policy analysis purposes, then a policy is normally modeled as a change of a certain, or multiple, parameters. The problem then is that a change in a parameter with no actual socio-linguistic meaning does not correspond to any actual policy. In some cases, vague and broad socio-linguistic interpretations of model parameters are offered, but the questions remains how actual policies can be quantified as changes in a certain model parameter. A frequently used parameter is the *status* or *prestige* of a language. Although this parameter corresponds to a socio-linguistic concept, it necessarily fails to map the complexity of this concept, since it is boiled down to a single number in the models. As the authors in Fernando *et al.* (2010) rightfully note: “what were the characteristics of a language having a prestige value, say 1.2, and what was the sociocultural condition corresponding to the difference between two languages having prestige values, say 1.2 and 1.3, respectively” (p. 50). As a consequence, they offer a model without a status parameter but with parameters that are – at least theoretically – measurable in the field. Models are therefore often bound to general relations of the form: *if A increases, then B increases*. A noteworthy exception is the approach taken in Sabourin & Bélanger (2015) and, more recently, in Houle & Corbeil (2017). Without any direct reference to the existing literature on language competition modeling, but informed by quantitative and qualitative socio-linguistic research, as well as literature on demolinguistic projections, Sabourin & Bélanger (2015) propose a microsimulation model to analyze language dynamics in Canada, purely based on available empirical information. Taking into account mother tongue and home language, the authors consider intergenerational language shift (mothers transmitting her own mother tongue) and intragenerational language shift (language used at home differs from mother tongue). Using age and group specific birth and death rates, a division of Canada in 13 regions and detailed (linguistic) information on migrants, the authors analyze different possible future scenarios for the linguistic composition of Canada (until the year 2046). The analysis of multiple future scenarios shall inform policy makers since, if adequately parameterized, the model can "realise virtual social experiments" (72). While rightfully emphasizing the role of migration, education is not included in the model directly, although education "tends to be the single most important channel of government intervention in the sphere of language" (Grin 2003, p. 17). Moreover, they only consider the mother for intergenerational language transmission instead of potentially mixed families, as done in Wickström (2005) and elsewhere.

In this essay, we offer a modeling approach that brings us closer to a practicable framework for dynamic policy evaluation. All model parameters are obtainable from empirical socio-linguistic data, which are already available today in some countries. As noted above, the model here comprises five factors that are crucial to the language dynamics in any (modern) society: 1) couple/family formation, 2) intergenerational language transmission, 3) language acquisition in formal education, 4) language learning by adults, including migrants and 5) migration. In contrast to Wickström (2005), language transmission is not modeled as a function of the strength of the language groups in the society and of the (relative) status of the minority language. Instead, it is assumed to be constant and shall be obtained from empirical data. This approach yields some advantages, but also limitations, in comparison to the original model. On the one hand, we can better simulate the actual linguistic environment in a specific context or society. Substituting the purely theoretical functional relations between model parameters and linguistic behavior in society with empirically obtained data allows us to analyze case scenarios in a more realistic fashion. On the other hand, this approach limits the time range of the model, since a constancy of parameters can only be assumed for limited time scales.[26] Furthermore, the role played by socio-economic elements of the linguistic environment is only indirectly captured by the current model, due to

[26] Appropriate time scales are an underestimated issue for many language dynamics models. Sometimes, a constancy of certain parameters is assumed for a hundred years or even longer, e.g. in Abrams & Strogatz (2003) and Minett & Wang (2008). In Wickström (2005), time scales are not even specified, so it remains unclear if the graphs presented there show time spans of 10, 100 or 1000 years. For the purpose of this essay, we restrict ourselves to time spans of at most 30 to 50 years.

the omission of the status parameter. In view of the problems with status-like parameters described above, we think that for our purposes the advantages outweigh the disadvantages.

4.2.1 Setting and Notation

We consider states or regions with two languages, a high status majority language H and lower status minority language L. Smaller languages like the ones of recent foreign migrants are omitted here. We group individuals according to their language repertoires. Three types of individuals are defined: monolinguals of the dominant language H, monolinguals of the minority language L and bilinguals. Treating an individual as a monolingual in the model does not imply that the individual only speaks one single language. It only means that the individual does not speak the other language well enough to count as a bilingual. Following Fernando *et al.* (2010, p. 53) we roughly define bilingualism as "the ability to function confidently in two languages, that is, the ability to have communicative competence in two languages". Hence, to be treated as a bilingual in the model, an individual has to have sufficient passive and active skills in both languages. When applying the model to a real life scenario, we often rely on self reported data. Accordingly, if people report to understand and speak both languages, they are taken as bilinguals. See the Basque example discussed below.

Throughout the essay, we use the following notation. X_H, X_L and X_B denote the proportion of H-monoglots, L-monoglots and bilinguals within the society. The two-dimensional vector $X := (X_H, X_L)$ fully describes the linguistic composition of the society, since $X_B = 1 - X_H - X_L$. Due to external migration, the population size can change over time. Therefore, we also work with absolute numbers. Let N_H be the total number of H-monoglots, N_L be the number of L-monoglots and N_B be the number of bilinguals. The total population size equals $\mathcal{N} := N_H + N_L + N_B$. Note that $N_H = \mathcal{N} \cdot X_H$, $N_L = \mathcal{N} \cdot X_L$ and $N_B = \mathcal{N} \cdot X_B$. Last, we introduce the three dimensional vector $N = (N_H, N_L, N_B)$.

4.2.2 Building the model

The language competition model describes in mathematical terms how N_H, N_L and N_B, i.e. the linguistic composition of the population, change over time as a result of the five processes mentioned above. We measure time t in years. $\mathcal{N}(t)$ is the overall population size at time t, while $N_R(t)$ is the overall number of people having language repertoire R, $R = H, L, B$. Furthermore, the vector $X(t) = (X_H(t), X_L(t)) := (N_H(t), N_L(t))/\mathcal{N}$ describes the relative linguistic composition of the population at time t (relative instead of absolute numbers).

We build up the model step by step. We start with a pure population model and add all of the five processes separately. By λ we denote the annual birth rate and by μ the annual death rate. It is assumed that birth and death rates are the same for all language repertoire groups. The model could easily be adjusted to cases of differing death and birth rates for the different language repertoire groups and family types (just substitute μ by μ_R and λ by λ_R in (4.47)). Omitting mobility for the moment, the overall population size changes according to $\dot{\mathcal{N}}(t) = (\lambda - \mu)\mathcal{N}(t)$. Note, to fully describe the dynamic system $\dot{N}(t)$, $\dot{N}_H(t)$ and $\dot{N}_L(t)$ are sufficient, since $N_B = \mathcal{N} - N_H - N_L$. Next, we add family formation and intergenerational language transmission to the basic population model.

Family formation

There are six possible couple/family types F: HH (two monolingual speakers of language H), HB (a monolingual speaker of H and a bilingual speaker), LL, LB, BB and HL. Family formation is conceptualized as the result of a random search and mating process. It is assumed that both adults are able to communicate in a common language. Hence, families of type HL are excluded. The distribution of family types depends on the linguistic composition of the society and on the geographical concentration of speakers of the minority language, see (4.42)-(4.46). Linguistic concentration is measured by the index of isolation, which measures the percentage of bilinguals in the geographical area of an average bilingual, see Morrill (2016). No linguistic concentration is represented by $C = 0$ and full concentration by $C = 1$. By $\psi(F; C, X)$ we denote the proportion of families of type F, given concentration C and the relative linguistic composition of the whole population X. The family formation process yields the following distribution of family types ψ:[27]

$$\psi(HH; C, X) = \big(C + (1 - C)X_H\big)X_H + (1 - C)X_H X_L, \tag{4.42}$$

$$\psi(HB; C, X) = 2(1 - C)X_H X_B, \tag{4.43}$$

$$\psi(LL; C, X) = \left(1 + C\frac{X_H}{1 - X_H}\right) X_L^2 + (1 - C)X_H X_L, \tag{4.44}$$

$$\psi(LB; C, X) = 2\left(1 + C\frac{X_H}{1 - X_H}\right) X_L X_B, \tag{4.45}$$

$$\psi(BB; C, X) = \left(1 + C\frac{X_H}{1 - X_H}\right) X_B^2. \tag{4.46}$$

Language transmission

It is assumed that families can transmit all the languages spoken by the parents to their children. In contrast, languages not spoken by the parents can not be transmitted in the family context. Hence, children growing up in HH or LL families will always be raised as H or L monolinguals. We denote the fraction of F-type families raising their children with language repertoire R by $q_R(F)$, $R = H, L, B$ and $F = HH, LL, HB, LB, BB$. As noted before, we assume that the language transmission values $q_R(F)$ are more or less constant over short- and medium-term periods of time and can be obtained from empirical data.

[27] See the Appendix for a justification of (4.42)-(4.46). For $C = 0$, we obtain the same formulas as Wickström (2005).

The product $q_R(F) \cdot \psi(F; C, X(t))$ yields the fraction of children coming from families of type F who are raised with language repertoire R. The basic population dynamics combined with family formation and language transmission can be described by the following three differential equations:

$$\dot{N}_R(t) = -\mu N_R(t) \; + \; \lambda \mathcal{N}(t) \sum_F q_R(F) \psi(F; C, X(t)), \tag{4.47}$$

$R = H, L, B$, where $\psi(F; C, X)$ is given by (4.42)-(4.46). The first summand represents the number of people with language repertoire R dying at time t. The second summand represents all the children raised with language repertoire R at time t.

Language learning in schools and by adults

Language learning at school is represented by parameters s_{R_1,R_2}, R_1, R_2 equal to H, L, B. The parameter s_{R_1,R_2} denotes the fraction of children entering school with language repertoire R_1 and leave school with repertoire R_2. It is assumed that children do not unlearn the majority language H. Furthermore, we assume that due to its dominance all children can speak language H by the end of school. Both assumptions make the model easier. Given appropriate empirical data, the model can also capture unlearning of H or failing acquisition of H by speakers of the minority language. For the simplified version considered here, we only have to specify $s_{H,B}$, $s_{L,B}$ and $s_{B,H}$. We use the notation

$$f_R(X) := (1 - s_{R,B}) \sum_F q_R(F) \psi(F; C, X) \; + \; s_{B,R} \sum_F q_B(F) \psi(F; C, X), \tag{4.48}$$

$R = H, L$. This notation shall help to improve readability. Adult language learning is represented by the parameters $a_{H,B}$ (H-monolinguals learning L and becoming bilingual). Since it is assumed that all adults speak the dominant language H, we do not consider learning of H by L monolinguals. If there is a relevant number of adult L monolinguals in the population, we would have to introduce a second parameter $a_{L,B}$ (L-monolinguals learning H). These parameters shall also be obtained from empirical data.

The model with family formation, language transmission, schooling and adult language learning is described by

$$\dot{N}_R(t) = -\big[\mu + (1 - \mu) a_{R,B}\big] N_R(t) \; + \; \lambda \mathcal{N}(t) f_R(X(t)). \tag{4.49}$$

Migration

The absolute number of people equipped with language repertoire R migrating at time t is denoted by $M_R(t)$. The total number of migrants is given by $\mathcal{M}(t) := M_H(t) + M_L(t) + M_B(t)$. It should be noted that in principle $M_R(t)$ could be negative, which would indicate net emigration of R's. Including migration, the overall population size $\mathcal{N}$ changes according to $\dot{\mathcal{N}}(t) = (\lambda - \mu)\mathcal{N}(t) + \mathcal{M}(t)$.

The final language competition model that includes all five processes is described by the three differential equations

$$\dot{N}_R(t) = -\big[\mu + (1-\mu)a_{R,B}\big]N_R(t) \,+\, \lambda\mathcal{N}(t)f_R(X(t)) \,+\, M_R(t), \qquad (4.50)$$

$R = H, L, B$. In the next paragraph, in which we apply the model to Basque and Spanish in the Basque Autonomous Communities in Spain, we assume constant relative external migration flow, that is, the number of migrants is a constant fraction of the population size. In mathematical terms this reads as $\mathcal{M}/\mathcal{N} \equiv const$. We denote this constant by ν.

4.3 Illustration of the model: the Basque Autonomous Communities

In this section we take a look at the Basque Autonomous Community (BAC) to illustrate how the model can be applied to a real case scenario. It should be noted right away that we do not present an in-depth analysis of the socio-linguistic reality of the BAC. For some of the model parameters, rather rough estimates are used. The aim of this section is for the reader to get an idea about how available data can be used to set up the model, and how the model can be used to produce future projections or different scenarios for future developments. We have chosen the BAC for a simple reason: it is one of the few regions of the world for which a wide range of language-related data were collected over several decades and are still collected today. Moreover, the BAC is a good example of a context in which a traditional minority language (Basque/Euskera) is widespread, in which the decline of the minority language could be decelerated and – partially – reversed successfully by language policies and which faced and still faces relevant migration movements.

The Basque Country (Euskal Herria) – the historical territory where the Basque language is spoken – comprises the BAC and Navarra in Spain and the Northern Basque Country (Iparralde) in France. Around 73% of the total population of the entire Basque Country live in the BAC, which is comprised of the three provinces Alava, Biscay and Gipuzkoa. Spoken by approximately one third of the population, Basque is a minority language within the BAC. The majority language is Spanish (Castilian).[28] On the territory of the BAC both languages have an official status, i.e. citizens have the right to know and use both languages. Moreover, the Spanish Constitution (1978) states that all Spanish citizens have a responsibility to know Spanish, cf. Zalbide & Cenoz (2008). Consequently, there are almost no Basque monolinguals in the BAC, since almost everybody can speak Spanish. Hence, Basque speakers are normally bilinguals. Unlike some other minority languages, Basque is not a language predominantly spoken in the rural countryside. Half of all Basque speakers live in cities.

[28]A few centuries ago, Basque was the majority language in all three parts of the Basque Country. For a historical overview on the centuries long decline of Basque and its revitalization since the 1960s see e.g. Trask (1997).

4.3.1 Data and model parameters

Most of the data used in the following originate from the Sociolinguistic Surveys[29] as well as from the 2001 Census. The Sociolinguistic Survey is a study conducted every five years, starting in 1991. It provides a variety of data on language for all three regions of the Basque Country. The survey covers inhabitants of the Basque Country aged sixteen and above, i.e. about 2.5 million people. In 2006, 7200 people were surveyed by using a structured closed-ended questionnaire. In 2011, 7900 people were surveyed, 4200 in the BAC. From the Sociolinguistic surveys we obtain data on the linguistic composition of the BAC for the years 1991, 1996, 2001, 2006, 2011 and 2016. The 2001 Census contains self-reported data on citizen's language knowledge, mother tongue and home language as well as other relevant information. We use 2001 as a reference year, that is, model parameters are estimated from data from around 2001. We have chosen 2001 for two reasons. Firstly, to estimate all the different model parameters, various data are needed. For most years within the time span 1991-2016, only some of the relevant data are available. In contrast, most of parameters can be estimated from studies from around 2001, and it is preferable to use data that all stem from one point in time rather than using data collected over 20 years. Moreover, we assume parameters to be constant over time. Since we do not only want to reproduce the development of the linguistic composition between 1991-2016, but also want to offer projections until 2040, it is reasonable to take data from a year close to the mean of 1991 and 2016.

Linguistic composition

We start with the linguistic composition of the BAC. We restrict ourselves to (self-reported) language competence in the two main languages, Spanish and Basque. The Sociolinguistic Survey differentiates between full bilinguals, passive bilinguals and non-Basque speakers. Full bilinguals speak and understand both languages well. Passive bilinguals "understand Basque although they do not speak it well" (Basque Government 2008, p. 17). In the current version of the model we only differentiate between monolinguals and bilinguals. Therefore, we only count full bilinguals as bilinguals, while passive bilinguals are counted as monolinguals. Moreover, we make the simplifying assumption that all non-Basque speakers speak Spanish well, which is the case for the vast majority of them.[30] Given this simplified classification in Bilinguals and Spanish monolinguals we obtain the numbers shown in Table 6.

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[29]Cf. Basque Government (2008), Basque Government (2013) and Gobierno Vasco *et al.* (2016). Data from the first four Sociolinguistic Surveys are made available online with the Language Indicator System of the Basque Country (EAS), see http://www1.euskadi.net/euskara_adierazleak/indice.apl

[30]In future extensions of the model we will consider a wider range of language skills and more than two languages. Although such extensions yield a more complicated and harder to handle mathematical model, the basic idea of the current model, as well as of its application to real-life cases, remain the same.

Year	1991	1996	2001	2006	2011	2016
H	75.9%	72.3%	70.6%	69.9%	68.0%	66.1%
B	24.1%	27.7%	29.4%	30.1%	32.0%	33.9%

Table 6: Linguistic composition of the BAC population aged 16 and above from 1991 to 2016. H beeing Spanish monolinguals and B being bilinguals.

Based on self-reported data on language skills in the Sociolinguistic Surveys. The BAC population aged 16 and above comprises about 1.8 million people.

Population dynamics and migration

The BAC is facing a similar situation as other European states: the birth rate is relatively low and the population is aging. Additionally, the BAC has a long history of inward and outward migration. In 2001, according to the Census, 27% of the BAC population were born outside the BAC. Many of them came from Navarra or the rest of Spain. In recent years the proportion of migrants from outside of Spain has increased dramatically, with many coming from south America. During the 1990s, the population size decreased, but since the beginning of the new millennium, numbers have been increasing slightly again, cf. Figure 12.

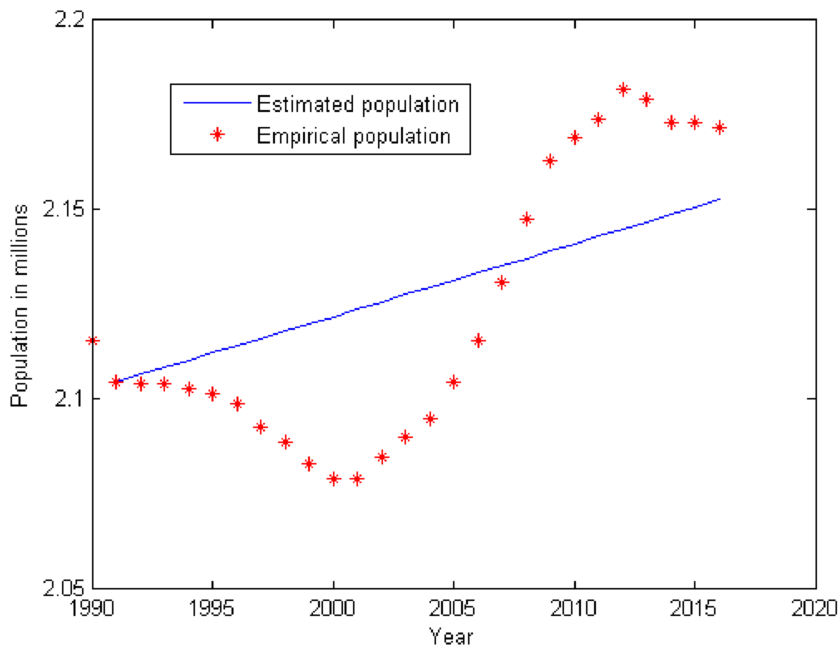


Figure 12: Estimated population (constant birth, death and migration rate) plotted against empirical population between 1990 and 2016.

Between 1991 and 2010 the annual birth rate increased from 7.7 births to 9.7 births per 1000 people, cf. Eustat (2014). Before 1990, the birth rate had been decreasing. For the purely illustrative purpose of this section, we assume a constant birth rate. As in the remainder of this section, we use 2001 as our reference year. In 2001 the birthrate was 8.5. Hence, we set λ to 0.0085. Due to the aging population, the death rate also increased slightly. As for the birth rate, we feed the model with the 2001 death rate. Hence, we set $\mu = 0.0088$.

Within the 20 year period between 1991 and 2011 the net migration rate changed significantly. During the 1990s, emigration outnumbered immigration, yielding a negative rate (-2.3 per thousand in 1990). Since 2000 the rate has been positive. The net migration rate reached its peak in 2007 (around 8 per thousand) and almost reached zero again in 2012, cf. Eustat (2014). Again, we work with the 2001 rate ($\nu = 0.0012$), which is slightly bigger than the average net migration rate for the 20 year period.

Family formation and concentration

In the model, the distribution of the five family types depends on the linguistic composition of the society in question, as well as on linguistic concentration. To estimate linguistic concentration, we once again use data from the Sociolinguistic Surveys that are made available online at the Language Indicator System of the Basque Country (EAS). There we find data on the linguistic composition of each of the 250 BAC municipalities. We use 2001 as a reference year.

To measure linguistic concentration we use the index of isolation. Given K municipalities, this index is given by $C = \sum_{k=1}^{K}(N_{B,k} \cdot N_{B,k})/(N_B \cdot N_k)$, where $N_{B,k}$ is the number of bilinguals in municipality k, N_k is the number of inhabitants of municipality k and N_B is the total number of bilinguals. This index measures the percentage of bilinguals in their own geographical area for the average bilingual speaker. $1 - C$, the index of exposure or interaction index, measures the extent to which average bilinguals are exposed to Spanish monolinguals.[31] For the BAC we obtain $C = 0.476$ and hence medium linguistic concentration.

Language transmission

Data on language transmission are obtained from Gobierno Vasco (2008), a study on the transmission of Basque in the BAC based on the 2001 Census. Children and young people between the ages of 2 and 29, as well as their parents, are surveyed. Only those children who still live with both of their parents are taken into account. The Census contains information on language skills and the mother tongue of children and their parents. It also contains information on language use at home but, in accordance with the model, we only consider parents' language skills.

To set up the model, we have to specify $q_R(F)$ – the proportion of all F-type families that bring up their children with language repertoire R – for all family types F and all language repertoires R. As was previously mentioned, we have

[31] For a detailed discussion of several concentration/segregation measures see e.g. Morrill (2016).

three language repertoires (L - monolingual Basque, H - monolingual Spanish and B - bilinguals)[32] and three relevant family types (HH, HB and BB). The data presented in Table 7 was obtained from from (Gobierno Vasco 2008, p. 53).

$R \setminus F$	HH	HB	BB
L	1.7%	35.1%	88%
H	94.8%	34.8%	5.1%
B	3.1%	30.1%	6.9%

Table 7: Language transmission in the BAC based on data from the 2001 Census. The table shows the mother tongue of the children based on their parents' linguistic competences or language repertoires.

At first glance, some the numbers in Table 7 might be surprising. In the first column, we can see that 1.7% of all HH-type families, i.e. families with two Spanish monolingual parents, raise their children (only) in Basque. Even more families with one Spanish monolingual parent and one bilingual parent, namely 35.1%, also raise their children (only) in Basque. This can partially be explained by our definition of Spanish monolinguals. Among these Spanish monolinguals are several people who reported to have a passive knowledge of Basque. They might also have some active knowledge in Basque, but are not fully bilingual.[33] For future research, a finer categorization of language skills should be applied. Despite the simple classification used here, empirical data are matched surprisingly well (see below).

Schooling

In the BAC students have to go through 6+4 years of compulsory education. Approximately 50% of all students go to public schools, while the other 50% attend private schools. With regard to languages, the theoretical goal of the Basque education system is that all students should learn Spanish and Basque. To achieve this goal, both languages are used either as the language of instruction or as a second language. There are three basic school models.

Model A: Instruction in Spanish with Basque taught as a second language.

Model B: Instruction in both languages.

[32]For the linguistic composition of the adult population (aged 16 and above) we only consider Spanish monolinguals and bilinguals. This is due to the assumption that at some point in their life every inhabitant of the BAC learns to speak the dominant language, Spanish, sufficiently to count as a bilingual. In case of the parents, we consider their language skills. For the children, we consider their mother tongue. Thus, we also take into account Basque monolinguals, since parents can communicate with their children only in Basque, so at a young age these children might be truly Basque monolinguals.

[33]In Gobierno Vasco (2008) passive bilinguals are those who can speak Basque with difficulties ("hablan en euskera con dificultad"). Full bilinguals can speak Basque well ("hablan bien en euskera").

Model D: Instruction in Basque with Spanish taught as a second language.

Enrollment in each of the three models has dramatically changed over time. At the beginning of the 1980s, almost 80% of pupils were enrolled in model A and model X (no Basque at all) schools. In 2006 less than 25% of all students were enrolled in model A schools, the rest being enrolled in model B and D schools. A study of the 2004/2005 school year found that 95.7% of all students in model B schools do not speak Basque at home, cf. ISEI-IVEI (2005). In model D this percentage is 63.2%.

To estimate the education-related model parameters – i.e. $s_{H,B}$, $s_{L,B}$ and $s_{B,H}$ – we not only need data on enrollment, but also data on school attainment with respect to language. Such data are provided in ISEI-IVEI (2005). In this study the Basque proficiency of students from model B and D schools at the end of compulsory education was surveyed. The study investigated whether or not students had reached a B2 level in Basque, using a written and an oral test. A pilot test showed that even the best model A students did not achieve a B2 level in Basque. For that reason, only model B and D schools were tested in the final study. Consequently, we assume that no model A students – most often native speakers of Spanish – leave school as bilinguals. Based on the written test, 32.6% of pupils from model B schools and 68% of those from model D school reached level B2. When taking oral test into account as well, the percentages are slightly lower. What is interesting here are the results for language use at home. If Basque is not spoken at home, 26.6% of model B students reach a B2 level. In model D schools this number is 47.5%. If Basque is spoken at home, the percentages are 47.5% in model B schools and 74.1% in model D schools.Moreover, Basque is not used at home by 95.7% of all model B students. In model D schools only 63.2% of all students do not speak Basque at home. From these figures one can see that "[m]any of the new bilinguals are Spanish-dominant bilinguals who are speakers of Basque as a second language and have learned Basque at school" (Zalbide & Cenoz 2008, p. 7)

Taking the enrollment data from our reference year 2001 in the different models,[34] and assuming that in model A schools we find no pupils who use Basque at home, we obtain the following rough estimates: Around 20% of all pupils who do not speak Basque at home (H-type children) leave school with a B2 level in Basque (and could thus be counted as bilinguals $\rightarrow s_{H,B} \approx 0.2$). Equivalently, we obtain $s_{L,B} = s_{B,B} \approx 0.76$ and $s_{L,H} = s_{B,H} \approx 0.24$.

Adult language learning

Estimating the rate at which adult Spanish monolinguals learn Basque ($a_{H,B}$) is difficult. To get a precise estimate we would have to evaluate longitudinal data on adults' language skills and language learning efforts. To our knowledge, such data are not yet available. Therefore, we are currently only able to present very rough estimates. As the aim of this section is to give the reader an idea on how

[34]Model A: $\approx 37\%$; model B: $\approx 10\%$; model D: $\approx 43\%$.

the theoretical model can be applied to a real life scenario, such rough estimates should be be sufficient. For an in-depth analysis of the language dynamics in the BAC more accurate data are needed. Naturally, the rate $a_{H,B}$ has a strong impact on the overall language dynamics. To illustrate that, consider a monolingual population with 1000 people. If every year 1% of the monolinguals learn a second language, all other language dynamics aside, then after 20 years around 180 people are bilingual. If, in contrast, every year 2% learn a second language, then after 20 years more than 330 people are already bilingual.

"Basque language teaching for adults is available through two very different routes in the BAC, via Department of Education Official Language Schools or via the euskaltegi (Basque language school) network under the aegis of HABE" (Gardner 2002, Appendix). According to Gardner (2002), about 5000 people were enrolled in Official Language Schools in 1999/2000 to learn Basque. During the same year, more than 40000 people enrolled in an euskaltegi.[35] Comparing these numbers with the total number of Spanish monolinguals during that time (about 1.26 million), we estimate that in this academic year around 3.5% of all Spanish monolinguals took a Basque language course. We count B2 speakers, and only consider people who reach the B2 level. If someone reaches C1 or C2, then we could argue that he/she was already bilingual before. In Gobierno Vasco (2003) we find that between 1993 and 2002 approximately 14.6% of the euskaltegi students reached a B2 level. Therefore, as estimate for $a_{H,B}$ we take $a_{H,B} = 0.146 \cdot 45000/1260000 \approx 0.005$. That is, we estimate that every year 0.5% of all H-monolinguals become bilingual due to adult language learning.

4.3.2 Analysis and projections

After having specified all the model parameters we can now run the model to (1) see whether the model outcome matches the actual data from our 20 year period and to (2) produce projections for the future. For that purpose the model was implemented in MATLAB, a widely used numerical computing environment and programming language. For the numerical model, the point of departure is the year 1991. All the model uses to calculate the projection are the linguistic composition at that initial year and the above specified model parameters. As can be seen in Figure 13, the projections match the empirical data remarkably well, despite the often rough model parameter estimates. The projections furthermore suggest that Basque will continue gaining speakers in the next 20 years. If the trend continues, then by 2040 approximately 40% of the BAC population will be bilingual.

It should be noted again that although the model starts from the initial 1991 linguistic composition and adequately projects the composition of the following years (1996-2016), most of the model parameters were estimated from data from

[35] It is worth noting that enrollment numbers started decreasing at the end of the 1990s, especially for A1 and A2 classes in Basque, cf. Gobierno Vasco (2003). This can be correlated with the increasing numbers of pupils in schools of model B and D.

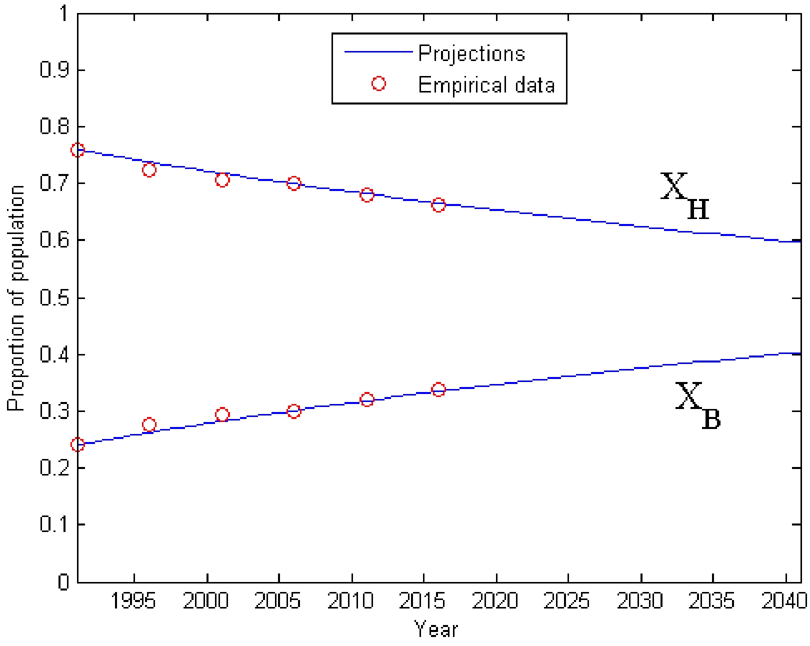


Figure 13: Language dynamics projections for the BAC compared to empirical data.

Projections (solid lines) are based on composition data from 1991 as well as the various estimations of model parameters for our reference year 2001. The circles show the actual fractions of Spanish monolinguals and bilinguals for the years 1991, 1996, 2001, 2006, 2011 and 2016.

around 2001. If parameters would have been estimated from 1991 data, then the projections might not have been that accurate. Unfortunately, we are not able to compare the projections for these two cases since not all of the relevant data are available for 1991. Projections only differ if parameters change over time. So what we can do instead is to examine how changes in model parameters affect the projections produced by the model. This is illustrated in Table 8 and Figure 14. In Table 8, we show how changes in a single parameter alter the distance between projections produced by the model and the empirical data we have for the years 1991, 1995, 2001, 2006, 2011 and 2016. We employ a simple quadratic distance measure:

$$dist = 100 \cdot \sqrt{\sum_{t=1991}^{2016} \left(X_H(t) - X_{H,emp}(t)\right)^2},$$

where $X_H(t)$ is the projected fraction of H-monolinguals at year t and $X_{H,emp}(t)$ is the observed fraction. For the original set of parameters, the distance is 1.93. As can be seen in Table 8, the parameters related to the majority language group ($s_{H,B}$ and a_{HB}) have the strongest effects. This is what one would expect. In the simple numerical example provided in subsection 4.3.1, we could already see the importance of adult language learning (a_{HB}). Table 8 also shows that changes in other parameters do not have a strong effect on the projections and that some changes even yield better results than our original parameter estimates.

original parameter	modified parameter	distance
$q_H(HB) = 0.348$ $q_B(HB) = 0.301$	$q_B(HB) = 0.5$ $q_H(HB) = 0.149$	1.82
$C = 0.47$	$C = 0.6$	1.84
$s_{L,B} = 0.76$, $s_{L,H} = 0.24$	$s_{L,B} = 0.9$, $s_{L,H} = 0.1$	2.27
$s_{H,B} = 0.2$	$s_{H,B} = 0.3$	3.16
$a_{H,B} = 0.005$	$a_{H,B} = 0.006$	3.57

Table 8: Distance between empirical observations and projections after changes in single parameters (1991-2016).

For the original set of parameters, the distance is 1.93. The last example ($a_{H,B} = 0.6\%$) is depicted in Figure 14.

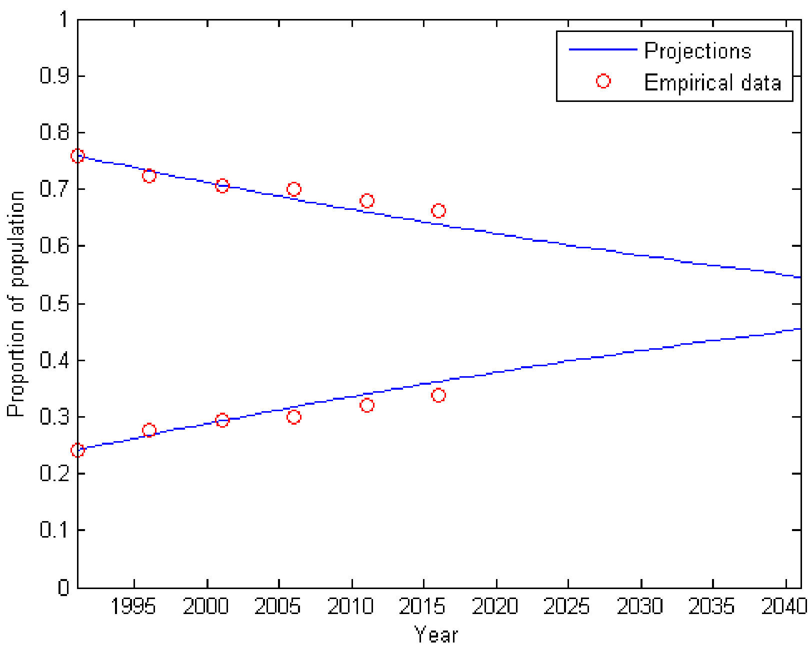


Figure 14: Language dynamics projections for the BAC compared to empirical data for $a_{HB} = 0.6\%$.

All other parameters are as above.

4.4 Dynamic language policy evaluation

Cost-Benefit analysis (CBA) and Cost-Effectiveness analysis (CEA) are two common methods to compare different policy options. In this section, we present a dynamic versions of CBA designed for the evaluation and comparison of language policies.[36] We are interested in language policies at the state or the regional level aiming at improving the status of the minority language. Policies can range from putting up bilingual street signs to printing bilingual bank notes to providing public services in the minority language. Another important example is acquisition planning, for example, teaching the minority language to children in schools or funding minority language classes for adults. For such policies, the overall benefits often depend on the number of beneficiaries, as do, in many cases, the costs of the policy. If policies specifically target speakers of the minority language, then, consequently, costs and benefits change over time as the linguistic composition of the population changes. The idea is therefore to utilize the information on changes in the linguistic composition that we gain from the language competition model in order to modify the standard CBA. The remainder of this section is devoted to the formalization and illustration of this simple idea.

4.4.1 Policies in the language competition model

In the introduction, language policies were defined as "a systematic, rational, theory-based effort at the societal level to modify the linguistic environment". Within the model framework, the linguistic environment is characterized by model parameters related to linguistic concentration, language transmission, education, adult language learning and migration. Consequently, language policies can be modeled as a change in one or several of these parameters. Some language policy measures of interest, such as putting up bilingual street signs or printing bilingual bank notes, might not affect one of the modeled elements of the linguistic environment directly. Nonetheless, they could have indirect effects. For example, increasing the visibility of the minority language in the public sphere could yield an extra incentive for adults to learn, maintain or transmit the minority language, and hence an increase in the respective parameters.

It is important to emphasize that we can only model the (expected) effect of a policy on one or several of the model parameters, rather than the policy itself. This shall be illustrated with two examples. First, consider the replacing of monolingual street signs by bilingual ones. Obviously, there are no street signs in our model. Hence, the model is not able to make a prediction about whether or not bilingual signs motivate more people to learn or maintain the minority language. If those who design the policy expect it to increase minority language learning rates by monolingual speakers of the majority language, then this expected increase is modeled as an increase in the respective parameter $a_{H,B}$. If the policy is expected to have no effect on any of the model parameters, then the parameters do not change. As a second example, consider the introduction of bilingual education programs. Such an acquisition planning policy directly aims to strengthen language skills in the minority language of pupils. Therefore, if effective, such a

[36] The same idea can be applied to obtain a dynamic version of CEA. We therefore only present the dynamic CBA.

policy changes the linguistic composition, or at least counteracts the decline of the minority language. But again, the education system is not modeled as such. What is modeled are its effects on the language skills of pupils at the end of compulsory education. The introduction of a bilingual education program can hence be translated into the model framework as an increase $s_{H,B}$ (the fraction of pupils entering school as H-monolinguals and leaving school as bilinguals) and/or a decrease of $s_{B,H}$ (the fraction of students entering school as bilinguals and (partially) losing their skills in the minority language until they leave school). The estimation of the effects of a policy has to be provided by experts from the respective fields, e.g. language education, or has to be drawn from *ex post* policy analysis of similar policies in comparable contexts.

In the following, we denote policies by p. Moreover, we denote the status quo, i.e. if everything stays as it is today, by p_0. For every policy p, including p_0, we have, as just explained, a set of parameters $param(p)$. If a policy does not change any of the parameters, then $param(p) = param(p_0)$. If policy p is applied and if we have an initial linguistic composition $N(0)$, then the linguistic composition $N(p;t) = (N_H(p;t), N_L(p;t), N_B(p;t))$ evolves according to

$$\begin{aligned} N_R(p, t+1) = N_R(p,t) &- \big[\mu + (1-\mu) a_{R,B}(p)\big] N_R(p,t) \\ &+ \lambda \mathcal{N}(p,t) f_R(p, X(p,t)) \; + \; M_R(p,t). \end{aligned} \tag{4.51}$$

Formula (4.51) is a discrete version of the differential equation (5.98), describing the evolution of the sizes of the language repertoire groups on an annual basis. The dependency of the model parameters on policy p is captured in this formula by $a_{R,B}(p)$, $f_R(p,\cdot)$ and $M_R(p,t)$. In principle, the birth and the death rate (λ and μ) could also depend on p.

As a first step of dynamic policy evaluation one has to derive the projections $N(p,t)$ for $t = 0, 1, ..., T$ and all the policies p that will be investigated. Final year T is the end of the time horizon during which one wants to evaluate the policy.[37]

4.4.2 Dynamic CBA

To compare benefits b and costs c of a policy p with CBA, both have to be specified. Benefits at the societal level are simply the sum of all individual benefits (or propensities to pay) of the policy in question. That is, $b(p) = \sum_i b^i(p)$, where $b^i(p)$ is the propensity to pay of individual i for a change from the status quo p_0 to policy p. To streamline the following presentation of static and dynamic CBA, we make the simplifying assumption that the benefit of a policy is the same for all individuals with the same language repertoire. In our two languages setting, the overall benefits of a policy measure p are then fully determined by the three quantities $b_H(p)$, $b_L(p)$ and $b_B(p)$, where $b_R(p)$ is the benefit of policy measure

[37]Final time T could also be set to infinity. Since parameters are assumed to be constant, a finite horizon is preferable.

p for an individual having language repertoire R. Depending on the policy to be analyzed, $b_R(p)$ can be a constant or change with the linguistic composition of the society, represented by the vector $N = (N_H, N_L, N_B)$.[38] Then, the aggregated benefit of policy p is given by

$$b(p, N) = N_H \cdot b_H(p, N) + N_L \cdot b_L(p, N) + N_B \cdot b_B(p, N). \tag{4.52}$$

If, for example, only monolingual speakers of the minority language (L's) benefit from policy measure p and if benefits do not depend on the linguistic composition, then the aggregated benefit of p is proportional to the number of L-monolinguals.

Next, we specify costs. When speaking of costs, we mainly think of public expenditures like costs for policy design, implementation costs and costs for monitoring and evaluation. In addition to these, the costs of a policy measure can come in the form of negative impacts of the policy for certain (linguistic types) of individuals. As for the benefits, we make the simplifying assumption that this second kind of costs, if there are any, are the same for every individual with the same language repertoire. Hence, these costs have the same form as the benefits in equation (4.52). For the first type of costs, we can distinguish costs that are independent of the linguistic composition (fixed costs c^f) and costs that vary with the number of people affected by the policy (variable costs c^v).[39] Typical fixed costs are costs for policy design and monitoring.[40] Since different fixed costs might emerge at different points in time, we allow c^f to depend on time, denoted by t. In the above example, variable costs would include salaries for teachers and expenditure for teaching materials for all the students subject to the policy. We assume that the variable costs are given by number of individuals subject to the policy multiplied by per capita costs. To summarize, costs at some time t are given by

$$c(p, t, N) = c^f(p, t) + c^v(p, N). \tag{4.53}$$

Denoting the per-capita variable costs for an individual with language repertoire R for a policy measure p by $c_R^v(p)$, we obtain

$$c(p, t, N) = c^f(p, t) + N_H \cdot c_H^v(p) + N_L \cdot c_L^v(p) + N_B \cdot c_B^v(p). \tag{4.54}$$

Both costs and benefits, are specified with respect to the existing *status quo*. Policy p is a potential improvement of the status quo, if societal benefits exceed societal

[38] Take, for example, a policy strengthening education in the minority language. If effective, such a policy reduces language loss among the minority language group. Moreover, being proficient in the minority language enables communication in that language with other speakers. Thus, it would be reasonable to assume that the communicative value of speaking the minority language and hence the individual benefit from the policy depend on the number of speakers of that minority language, and hence that the individual benefits are changing endogenously. Therefore, not implementing such a policy can negatively affect acquisition of the minority language as well as the future evaluation of policies in favor of the minority language by its speakers.

[39] Additionally, costs of a policy could depend on the geographical distribution of its beneficiaries, see e.g. Wickström *et al.* (2018).

[40] Other fixed costs could, e.g., emerge from the design of text books and other learning material in the minority language.

costs, i.e. $b(p,N) > c(p,N)$. It is an improvement in the allocative sense of Pareto efficiency: those who gain from policy p could, theoretically, fully compensate those who lose and would still be better off. If different policy options are available for which benefits exceed costs, the policy maker should, from a purely allocative point of view, choose the one with the highest difference between costs and benefits.[41] To also account for costs and benefits in the future, we have to calculate the net present value (NPV) of a policy. Measuring time in years and assuming an annual discount rate r, the NPV of language policy p is given by

$$NPV(p) = \sum_{t=0}^{T} \frac{b(p,N) - \big(c_f(p,t) + c_v(p,N)\big)}{(1+r)^t}, \tag{4.55}$$

where T is some final time horizon.

Although costs and benefits at different points in time can be accounted for in the classical discounted cost-benefit analysis, potential changes in the number of people affected by the policy are neglected.[42] Such changes can be caused by policy itself or can be the result of a more general trend, for example, the steady decline of the minority language that has been observed over the past few decades. If these dynamics are neglected, then costs and/or benefits might be over- or underestimated. Therefore, for the dynamic cost-benefit analysis, we work with the projections $N(p,t)$, $t = 0, 1, ..., T$, provided by the language competition model, cf. formula (4.51), instead of the constant numbers used in the classical CBA. Accordingly, we obtain the dynamic net present value

$$DNPV(p) = \sum_{t=0}^{T} \frac{b(p,N(p,t)) - \big(c^f(p,t) + c^v(p,N(p,t))\big)}{(1+r)^t}, \tag{4.56}$$

where the initial $N(0)$ is given and where $N(p;t)$ evolves according to (4.51).[43] The DNVP can then be used as the NPV in standard CBA to decide whether aggregated benefits of a policy exceed aggregated costs or to rank different policy options according to their DNPV.

As was shown for CBA, a dynamic version of CEA can be obtained by using the projections derived from the language dynamics model instead of working with stationary population.

4.4.3 Illustration: dynamic CBA

To illustrate the computation of the dynamic NPV of a policy measure, we use our earlier numerical example: the Basque Autonomous Communities (BAC).

[41] In practice, compensation payments are often not or only partially feasible. For a more detailed discussion of allocative efficiency and compensation payments see e.g. Boardman *et al.* (2010, p. 26-33) and Wickström (2013).

[42] This is why in formula (4.55) the N does not depend on time, it equals $N(p,0)$ in the dynamic notation for all years.

[43] A similar approach was proposed in Wickström (2013), but without offering a specific model for the language dynamics.

The parameters of the *status quo* were provided in subsection 4.3.1. We assume that in our reference year, 2001, policy makers consider launching a new television channel to extend the range of the public broadcasting service.[44] This new channel shall offer a program in Basque for children and young people in the BAC.[45] It is assumed that annual fixed costs are 5 million Euro. [46] Neglecting revenue from advertising, costs do not depend on the number of people watching the channel. Hence, we set $c^f(p,t) = 5,000,000$ and $c^v(p,t) = 0$. We assume that the policy makers assign an annual per capita benefit of $b_B(p)$ Euro, and that only speakers of Basque profit from this channel. Hence, $b_H(p) = 0$. As potential viewers, children can profit from the channel directly. Adult speakers of Basque, although probably not consuming the new channel themselves, can still value its existence, since it strengthens the Basque language among the younger generation. Considering a twenty year time horizon ($T = 20$) and a 5% discount rate ($r = 0.05$), we obtain overall discounted costs of about 65 million Euros. In the classical CBA, discounted benefits are $b_B(p)$ multiplied by 6.7 million Euros. Including the language dynamics, discounted benefits are estimated to be $b_B(p)$ multiplied by 7.7 million Euros. The difference is due to the projected increase of Basque speakers. Taking, for example, an annual per capita benefit of 10 Euros, we get $NPV(p) = 150,00$ and $DNPV(p) = 1.2$ million. Despite obtaining this notable difference, we do not assume a positive effect of the policy on Basque revitalization so far. Assuming such a positive effect would make the dynamic NPV even higher. If annual per capita benefits are only 9 Euros, then we even get $NPV(p) = -500,000$ and $DNPV(p) = +400,000$. In the latter example, the classical CBA suggests not implementing the program, while the dynamic CBA suggests the opposite. This is due to the fact that the classical CBA does not account for conceivable changes in the linguistic structure of the population.

4.4.4 Efficiency, equality and normative implications

As the purpose of this section is to illustrate how our language dynamics model can be employed to enhance existing policy analysis techniques for the evaluation of language policies, we only present a simple CBA with homogeneous propensities to pay within the language repertoire groups combined with the potential

[44] In 1982/83, ETB 1, the first television channel of the BAC's public broadcasting service EiTB was established. It provides programs exclusively in Basque. In 1986, ETB 2 was launched, a channel broadcasting in Spanish.

[45] In 2008, such a channel started broadcasting, called ETB 3. Here, we do not provide an evaluation of the introduction of ETB 3. The aim of this subsection is to use the numerical data we have found for the BAC to illustrate the dynamic cost-benefit analysis. The introduction of an ETB 3 like channel in 2001 shall function as an arbitrary but realistic example.

[46] Since no separate budget information is available for ETB3, we derived this estimate from EiTB budget information for 2007 and 2009. Between 2007 and 2009, the overall ETB budget increased by about 10 million Euros, mainly due to higher personnel and production costs, cf. Casado del Río & Miguel de Bustos (2015). One reason for this increase was the introduction of the new channel ETB3 in 2008. Therefore, we estimate annual additional costs for ETB 3 to be around 5 million Euros. For an in depth policy analysis, more detailed budget estimates are necessary.

Pareto efficiency decision rule. Both the analysis and the decision rule can be extended and adapted in several ways without affecting the basic dynamic policy analysis idea. For example, in standard CBA only efficiency is taken into account: as long as one person is better off after the introduction of a policy and nobody loses, a policy is deemed to be an improvement. The distributive effects of the policy are irrelevant in this regard. If inequalities matter – if, for example, policy makers want to support the least well-off–, then different (groups of) individuals can be weighted differently in the analysis, e.g. in a welfare function. If we want to achieve actual and not just potential Pareto improvements, then we could restrict the analysis to a politically and institutionally feasible set of transfer payments. Moreover, we omitted questions such as who is actually bearing the costs of the policy in question (only the minority language community or the entire society) or whether the costs and benefits for future generations should be accounted for by the analysis. Such aspects related to distributive effects of language policies, equality and feasibility can be included in the analysis, but we are not going into this detail here.[47] One way in which our approach is restricted is linked to the operationalization of benefits. To account for future benefits, we have to specify how they alter with changes in the linguistic environment. In the simplest cases, individual benefits of a policy are homogeneous within the language repertoire groups and constant, i.e. $b_R(p, N) = b_R(p) \equiv const$, which is a reasonable assumption for restricted time horizons. In this case, overall benefits are proportional to the number of beneficiaries. If individual benefits depend on the number of speakers of the minority language – as in the example in footnote 38 –, then this dependence has to be formalized, for example by a specification of $b_R(p, N)$ as a function of the vector N. An assessment of individual benefits or propensities to pay ($b^i(p)$ for every individual i or at least for a representative sample), for example by contingent valuation, can be problematic, since at the time of assessment ($t = 0$) we do not have any information of propensities to pay for the same individuals at later times or even for the next generation of individuals.[48]

Concerning general normative statements, the approach presented here is limited. The underlying language dynamics model provides a computable– if sufficient data are available – approximation of actual macro level language dynamics. In combination with language policy analysis methods, we get an applicable tool to analyze specific policy options for a specific state or region at a specific time. From an analysis with this tool we cannot derive any general normative implications, such as whether or at which threshold a minority language should be supported. The tool is designed for statements of the kind: *policy p in favor of language l should (not) be implemented* or *policy p is to be favored over policy q*. If the outcome of the analysis is that a policy in favor of the minority language should not be implemented, then this could clearly weaken this language.[49] In

[47]Most of these aspects are well illustrated and discussed in Wickström (2013).

[48]See e.g. Wickström *et al.* (2018) for a discussion of feedback mechanisms, endogenity, indirect effects and other difficulties when evaluating individual benefits of a language policy.

[49]"It can be efficient to introduce fewer minority rights than a simple static analysis would imply. We can find an "efficient discrimination" of minorities" (Wickström 2013, p. 336, emphasis in the original).

many cases, a normative assessment at a more general level is exogenous. This is particularly apparent in CEA, when desirable outcomes and effectiveness have to be defined before the actual numerical analysis. For CBA, the decision maker could be the person who assigns the values of $b_R(p)$ administratively or who decides on the weights given to the different language groups. Since the policy makers' decisions are part of the political process and are affected by voting behavior, it can be argued that they indirectly reflect actual propensities to pay.

4.5 Conclusions and outlook

In this essay we presented a novel framework for the evaluation of language policy measures. This framework was motivated by two observations. On the one hand, classical policy evaluation techniques, such as cost-benefit analysis or cost-effectiveness analysis, do not consider changes in the linguistic composition of the population affected by a policy measure. Hence, classical techniques potentially overstate or undervalue the actual benefits and/or costs of a policy measure. On the other hand, existing language dynamics models that aim to describe and sometimes explain changes in the linguistic composition are only of limited use for the analysis of specific language policy measures. Due to their abstract nature, most of these models are only suitable for the derivation of abstract or general results and policy recommendations. To facilitate the evaluation of specific policy measures for a specific context in a dynamic fashion, we proposed a new language competition model and illustrated how this model can be applied to improve existing policy evaluation techniques.

The model that takes into account five processes that are pivotal for changes in the linguistic composition of a population: population dynamics (birth, death and migration), family formation, language transmission, language education and adult language learning. In contrast to the vast majority of language dynamics models available in the literature, the proposed model operates on model parameters that can be estimated from empirical data. At the same time, the model is designed in a general fashion so it can be applied to states or regions with two languages, as long as the necessary data are available. Parameter estimation and the application of the model are illustrated by the case of Basque and Spanish in the Basque Autonomous Communities in northern Spain. Starting in 1990, projections for the (relative) numbers of Spanish monolinguals and bilinguals were derived for 50 years. Although parameters were derived from data from a single year (2000-2001) and kept constant for the entire time period, the projections match empirical data on the number of speakers from 1990 to 2016 remarkably well. To test the strength of the model beyond the Basque case, further empirical evidence for other contexts is needed.

Due to the neglect of changes in the numbers of beneficiaries and costs in language policy analysis, we proposed a combination of established policy evaluation techniques and the new language dynamics model. A dynamic version of the classical cost-benefit analysis was outlined. This essay illustrated how a non-dynamic cost-benefit analysis might underestimate the long-term benefits of a policy measure from which speakers of a growing minority population profit.

The aim of this essay was to show how established techniques used for language policy evaluation can be enhanced by explicitly taking into account language dynamics. While only illustrated for cost-benefit analysis, the basic idea can also be applied to cost-effectiveness analysis or other evaluation techniques. This is one reason why we consider this essay to be a starting point for further research on dynamic language policy analysis. Furthermore, the underlying language dynamics model can be extended to yield a more realistic depiction of the complex reality at hand. For example, one could consider different age groups, regions within the territory or more than two language groups. Moreover, certain model parameters could be different for different age or language groups (e.g. birth and death rates) or could change over time. All such extensions would make the model more realistic, but harder to handle. Although the model would become more complicated, the basic principles of the model itself as well as of the dynamic policy evaluation approach would remain the same. Furthermore, for all of these extensions, one would need even more empirical data on language knowledge, use and transmission, which are already scarce for most contexts, the Basque Autonomous Communities being an exception. Therefore, the model can be used to make an additional argument for a more detailed collection of language-related data in census-like surveys.

5 Extensions with generations, cohorts and multiple minority languages

5.1 Introduction

In Templin (2018),[50] we developed a simple language dynamics model with one majority language and one minority language. The population is divided into three language groups: monolinguals of the majority language, monolinguals of the minority language, and bilinguals. The model describes how the sizes of the three language groups change over time. It treats all individuals of a language group equally, independent of their age. In this essay, we refer to this model with one minority language and a single age group as a 1G-1L model.[51] In Templin (2019),[52] we argued that language dynamics models such as the 1G-1L model can be used to analyze costs and benefits of certain language policies. If language policies only target a certain age group within the population, e.g. pupils, or if there are multiple relevant language minorities within the society and a policy is only directed at one minority, the 1G-1L model might not be sufficiently complex to analyze such policies satisfactorily. Therefore, we present here two possible extension of the 1G-1L model.

The first extension introduces an age dimension.[53] We consider two models:

- In the **3G-1L model**, we consider three generations: children, young adults and old adults.
- In the **10G-1L model**, we divide the population into 10 age groups or cohorts.

The second extension is concerned with the case of multiple relevant minority languages.[54]

- In the **1G-nL model**, we consider the case of a majority language and n minority languages, with $n \geq 1$.

Before developing the extensions, we present a slightly generalized version of the basic 1G-1L model in Section 5.2. Thereafter, we discuss one possible way to extend the 1G-1L model to account for several age groups. In Section 5.4, we introduce an extension of the 1G-1L model to n minority languages, $n \geq 1$. Last, we compare the extensions to the original 1G-1L model numerically by

[50] Section 3 in this thesis.

[51] In the general introduction and the general conclusions of this thesis I refer to this model as model E^1.

[52] Section 4 in this thesis.

[53] In the general introduction and the general conclusions of this thesis I refer to this extension as model E^2_{mG}.

[54] In the general introduction and the general conclusions of this thesis I refer to this extension as model E^2_{nL}.

considering two empirical cases studied in Templin (2018) (English and Spanish in the United States) and in Templin (2019) (Spanish and Basque in the Basque Autonomous communities in Spain).

5.2 Model with one generation and one minority language (1G-1L)

In this section we revisit the basic model with two languages and without age-differentiation (1G-1L) presented in Templin (2018). The model describes how the linguistic composition of a population changes over time. The linguistic composition is determined by the distribution of language repertoires throughout the population. In the 1G-1L model, three repertoires R are relevant: monolinguals in a dominant language H, monolinguals in a minority language L and bilinguals. To obtain a realistic but manageable mathematical model, we take into account four key processes:

- **Population dynamics.** This comprises births, deaths, and inward and outward migration.
- **Intergenerational language transmission.** Adult individuals form couples, have children and transmit one or all of the languages they speak to the next generation.
- **Language education.** Pupils are educated in certain language(s) and learn additional languages in school.
- **Adult language learning.** Adult individuals learn additional languages or forget languages they do not use.

These processes are affected by the linguistic environment and shape the linguistic environment at the same time. To formulate a mathematical model, we first operationalize the linguistic environment. Thereafter, we formalize the four processes above and specify how they depend on the linguistic environment.

5.2.1 Linguistic environment

The linguistic environment is a theoretical construct. "It subsumes in an extensive (but obviously not exhaustive) fashion all the relevant information about the status, in the broadest sense of the word, of the various languages present in a given polity at a certain time" (Grin & Vaillancourt 1997, p. 49). For our language dynamics model, we consider five dimensions of the linguistic environment:

1. **Linguistic composition**. For the 1G-1L model, the linguistic composition of a population is determined by the number of adult monolinguals of the dominant language H, N_H, the number of adult monolinguals of the minority language L, N_L and the number of adult bilinguals, N_B. The total

population size is $N = N_H + N_L + N_B$. We also consider the fractions $X_R := N_R/N$, $R = H, L, B$.

2. **Linguistic concentration**. Concentration of speakers of the minority language is measured by a single concentration parameter $C_1 \in [0,1]$. For $C_1 = 0$ speakers of L are distributed equally throughout the territory, and for $C_1 = 1$ all speakers of L are concentrated in certain areas. To derive C_1, we use the index of dissimilarity, which can be interpreted as the proportion of speakers of L who have to move such that all areas have the same proportion of speakers of L, see Morrill (2016).[55] It is derived as follows. Assume that the territory can be subdivided in K regions. For $k = 1, ..., K$, N_k denotes the number of people living in region k, $N_{H,k}$ is the number of H monolinguals in region k, and $N_k - N_{H,k} = N_{L,k} + N_{B,k}$ is the number of speakers of L in region k. We set
$$C_1 := \frac{1}{2}\sum_{k=1}^{K}\left|\frac{N_{L,k}+N_{B,k}}{N_L+N_B} - \frac{N_{H,k}}{N_H}\right|. \tag{5.57}$$
Moreover, we consider concentration of monolingual speakers of L with respect to bilinguals. Analogously to C_1, we introduce
$$C_2 := \frac{1}{2}\sum_{k=1}^{K}\left|\frac{N_{L,k}}{N_L} - \frac{N_{B,k}}{N_B}\right|. \tag{5.58}$$
By C we denote the vector $C = (C_1, C_2)'$. Note, in Templin (2018) we only considered C_1, implicitly assuming $C_2 = 0$.

3. **Status of both languages**. The relative status of a language in society comprises two aspects that are weighted to get a single status parameter S:

   (a) **Socio-economic status**. We consider the average socio-economic status of H-monolinguals, $\overline{S}_{SE}(H)$, and the one of speakers of L, that is L-monolinguals and bilinguals, $\overline{S}_{SE}(L)$. We are interested in the socio-economic advantage individuals gain from speaking a certain language and hence consider the relative status $S_{SE}(H) = \overline{S}_{SE}(H)/(\overline{S}_{SE}(H) + \overline{S}_{SE}(L))$ and $S_{SE}(L) = 1 - S_{SE}(H)$.

   (b) **Institutional status**. The institutional/official status of a language is determined by the number of domains from a given list of domains a language can be used in. For a total number of D domains, the official status of L, $\overline{S}_{OF}(L)$, has as possible values $\{0, 1/D, 2/D, ..., 1\}$. The relative official statuses are given by $S_{OF}(L) = \overline{S}_{OF}(L)/(\overline{S}_{OF}(H) + \overline{S}_{OF}(L))$ and $S_{OF}(H) = 1 - S_{OF}(L)$.

   To obtain a single status parameter $S(L)$, we weight the socio-economic and the institutional aspect by weight α_{SE} and α_{OF}:
$$S(L) := \alpha_{SE} \cdot S_{SE}(L) + \alpha_{OF} \cdot S_{OF}(L). \tag{5.59}$$

[55]For alternative concentration measures see also Morrill (2016).

By construction, $S(L) \in [0,1]$ and $S(H) = 1 - S(L)$. The status variable $S(L)$ is a measure for the relative disadvantage of speakers of L in the society. Since H is the dominant language, we normally have $0 < S(L) < 1/2 < S(H) < 1$.

4. **Language education policy**. It is assumed that language education policies determine which languages are learned in schools. These policies are exogenous. Policies are not modeled as such, but only their effects on language learning. In the model, this is represented by parameters s_{R_1,R_2}, $R_1, R_2 = H, L, B$, the fraction of children entering school with language repertoire R_1 and leaving school with language repertoire R_2. We make the simplifying assumptions that $s_{H,L} = s_{B,L} = s_{L,L} = 0$, since H is the dominant language, and that $s_{B,H} > s_{L,H}$.

5. **Adult language learning support**. The acquisition of the majority language can be supported by language policy measures. This is reflected by a policy parameter $v_H \in [0,1]$. The higher v_H, the more the acquisition of H by L-monolinguals is supported. Similarly, support for the acquisition of the minority language is represented by a parameter v_L.

6. **Population statistics**. The birth rate λ and the death rate μ affect the speed at which language change occurs. Moreover, migration numbers affect the inflow and/or outflow of speakers of H and L. Throughout the essay, M denotes the annual net number of migrants (for negative M we have net emigration). M_H, M_L and M_B are the number of H monolingual, L monolingual and bilingual migrants, $M = M_H + M_L + M_B$.

Now that we operationalized the linguistic environment, we come back to the four key processes that drive the language dynamics in our model. Population dynamics is already described by the birth and death rates as well as by the migration numbers. Similarly, language education is modeled by the exogenous parameters s_{R_1,R_2}, $R_1, R_2 = H, L, B$. This leaves us with intergenerational language transmission and adult language learning. Their dependence on the linguistic environment is modeled in the following two subsections.

5.2.2 Family formation

The formation of families is modeled as a 2-step random search and mating process. In step 1, individuals meet randomly and form couples. In step 2, some couples successfully form families (produce offspring). Others are unsuccessful due to communication barriers and split up again. Who meets and forms couples depends on the distribution of speakers as well as on linguistic concentration. Which couples become families depends on the linguistic composition of the couple. After step 2, some individuals are already part of a family. All the remaining individuals start a second round of couple and family formation. This is repeated until all individuals are part of a family. Given the three language repertoire types H, L and B, there are six possible couple or family types F:

- $F = HH$: two H monoglots
- $F = HL$: one H monoglot and one L monoglot
- $F = HB$: one H monoglot and one bilingual
- $F = LL$: two L monoglots
- $F = LB$: one L monoglot and one bilingual
- $F = BB$: two bilinguals

We make the simplifying assumption that the population size N is large and even, and that the population consists of $N/2$ female and $N/2$ male individuals. This results in $N/2$ families at the end of the formation process. We assume that language repertoires are distributed equally among males and females. For every family type F, Ψ_F denotes the fraction of the $N/2$ families that are of type F. In the following we derive formulas for the fractions Ψ_F.

5.2.2.1 Family formation - monolingual case

To develop a mathematical model for family formation with mono- and bilinguals as well as two dimensions of concentration, we start with the easier case where there are no bilinguals ($N_B = 0$). In this simple case, there are only three relevant couple/family types, namely HH, HL and LL. To derive formulas for Ψ_{HH}, Ψ_{HL} and Ψ_{LL}, we first model couple formation. The probability, that a certain type of couple forms, depends on the distribution of speakers as well as on the linguistic concentration C_1, as defined in (5.57).

Let us start with one randomly chosen pair consisting of a female Y and a male Z. Furthermore, let R_1 and R_2 be two language repertoires. We have

$$\mathbb{P}[Y = R_1, Z = R_2] = \mathbb{P}[Y = R_1]\mathbb{P}[Z = R_2 \,|\, Y = R_1],$$

where $\mathbb{P}[\cdot\,|\,\cdot]$ denotes the conditional probability. To shorten notation we write $\mathbb{P}[R_1]$ instead of $\mathbb{P}[Y = R_1]$ and $\mathbb{P}[R_2 \,|\, R_1]$ instead of $\mathbb{P}[Z = R_2 \,|\, Y = R_1]$. The first factor on the right hand side, $\mathbb{P}[R_1]$, depends only on the fraction of people in the population with language repertoire R_1. Hence,

$$\mathbb{P}[R_1] = X_{R_1}. \tag{5.60}$$

For $C_1 = 0$, L-speaking individuals are distributed equally througout the territory. This translates to

$$\mathbb{P}_{C_1=0}[R_2 \,|\, R_1] = \mathbb{P}[R_2] = X_{R_2}. \tag{5.61}$$

For $C_1 = 1$, minority speakers only form couples with other minority speakers. Hence,

$$\mathbb{P}_{C_1=1}[R_2 \,|\, R_1] = \begin{cases} 1 & : \quad R_1 = R_2 \\ 0 & : \quad R_1 \neq R_2 \end{cases}. \tag{5.62}$$

To model probabilities for C_1 between 0 and 1, we use linear interpolation. Choosing $R_1 \neq R_2$, this yields

$$\mathbb{P}[R_1 \mid R_1] = C_1 + (1 - C_1)X_{R_1}, \tag{5.63}$$
$$\mathbb{P}[R_2 \mid R_1] = (1 - C_1)X_{R_2}. \tag{5.64}$$

Based on these conditional probabilities, we can derive the probabilities that a certain kind of pair is chosen. If $R_1 \neq R_2$, then a pair of type R_1R_2, is obtained either by $Y = R_1$; $Z = R_2$ or by $Y = R_2$; $Z = R_1$ (for the family type we do not take the gender of the parents into account). Hence, the probability of obtaining a pair of type R_1R_2 is given by $\mathbb{P}[R_1R_1] = \mathbb{P}[R_1]\mathbb{P}[R_1|R_1]$, and $\mathbb{P}[R_1R_2] = \mathbb{P}[R_1]\mathbb{P}[R_2|R_1] + \mathbb{P}[R_2]\mathbb{P}[R_1|R_2]$, if $R_1 \neq R_2$. This yields the following distribution of couple types, which are denoted by ψ_F for type F:

$$\psi_{HH} = X_H^2 + C_1X_HX_L \tag{5.65}$$
$$\psi_{HL} = 2(1 - C_1)X_HX_L \tag{5.66}$$
$$\psi_{LL} = X_L^2 + C_1X_HX_L. \tag{5.67}$$

To estimate the distribution of family types, Wickström (2005) takes a different approach. He offers a model without any concentration parameter – implicitly assuming $C_1 = 0$ – in which couples of type HL have a lower probability to become a family. All unsuccessful couples split up again, and a second round of couple and family formation begins. Somewhat surprising, he obtains similar results. If HH and LL couples have success probability 1 and HL couples have success probability ρ, Wickström (2005) derives formulas (5.65)-(5.67) with $C_1 = 1 - 2\rho/(1 + \rho)$.

As noted earlier, we also apply a two-step model for family formation. The family type distribution is denoted by Ψ. It is easy to see that if all couple types have the same success probability, then the distribution of family types equals the distribution of couple types, that is $\Psi_{HH} = \psi_{HH}$, $\Psi_{HL} = \psi_{HL}$ and $\Psi_{LL} = \psi_{LL}$, where the ψ_F are given by (5.65)-(5.67). The formulas are more complicated if linguistic concentration C affects couple formation in step 1 and if, additionally, communication barriers result in lower success probabilities p for HL couples.

Theorem 5.2.2.1. *If couples form according to* (5.65)-(5.67)*, we obtain the family type distribution*

$$\Psi_{HH} = X_H^2 + (1 - D(1 - C_1))X_HX_L \tag{5.68}$$
$$\Psi_{HL} = 2D(1 - C_1)X_HX_L \tag{5.69}$$
$$\Psi_{LL} = X_L^2 + (1 - D(1 - C_1))X_HX_L, \tag{5.70}$$

for $D = 1 - (1 - p)(1 + C_1)/(2 - (1 - p)(1 - C_1)$.

A proof is provided in the Appendix. We have $D = 0$ for $p = 0$ and $D = 1$ for $p = 1$. As one would expect, for $p = 0$ we obtain $\Psi_{HH} = X_H$, $\Psi_{LL} = X_L$ and $\Psi_{HL} = 0$.

5.2.2.2 Bilingual case

Previously, we considered the monolingual case to illustrate how we model family formation. In this section, we also take into account bilinguals. As before, the parameter C_1 is a measure for the concentration of (mono- and bilingual) speakers of L. As in the monolingual case, we start with couple formation. The population can be divided into speakers of L, $X_L + X_B = 1 - X_H$, and monolingual speakers of H. In view of (5.65)-(5.67), we set

$$\begin{aligned} \psi_{HH} &= X_H^2 + C_1 X_H(1 - X_H) &(5.71)\\ \psi_{HL} &= 2(1 - C_1)X_H X_L &(5.72)\\ \psi_{HB} &= 2(1 - C_1)X_H X_B &(5.73)\\ \psi_{LL} + \psi_{LB} + \psi_{BB} &= (1 - X_H)^2 + C_1 X_H(1 - X_H). &(5.74) \end{aligned}$$

We also consider the possibility that L-monolinguals are geographically concentrated with respect to bilinguals, measured by C_2. One reason for such concentration could be that L-monolinguals are newcomers (migrants), residing in specific areas, while bilinguals already spent some years in the new environment or were born there, and are spread out over the entire territory. For maximal concentration, i.e. $C_2 = 1$, the fraction ψ_{LB} is zero. For $C_2 < 1$ we derive ψ_{LB} similar to the monolingual model. Let $Y_L = X_L - \psi_{HL}/2 = X_L(1 - (1 - C_1)X_H)$ denote the fraction of the population that is monolingual in L and not in a couple with an H-monolingual. By the same token, consider $Y_B = X_B - \psi_{HB} = X_B(1-(1-C_1)X_H)$. Let $Y := Y_L + Y_B = (1-X_H)^2 + C_1 X_H(1-X_H)$ denote the fraction of couples that are of type LL, LB or BB. In view of (5.65)-(5.67), we set

$$\begin{aligned} \psi_{LL} &= Y\left(\frac{Y_L^2}{Y^2} + C_2\frac{Y_L}{Y}\frac{Y_B}{Y}\right) &&= \left(1 + C_1\frac{X_H}{1 - X_H}\right)\left(X_L + C_2 X_B\right)X_L &(5.75)\\ \psi_{LB} &= Y\left(2(1 - C_2)\frac{Y_L}{Y}\frac{Y_B}{Y}\right) &&= 2\left(1 + C_1\frac{X_H}{1 - X_H}\right)(1 - C_2)X_L X_B &(5.76)\\ \psi_{BB} &= Y\left(\frac{Y_B^2}{Y^2} + C_2\frac{Y_L}{Y}\frac{Y_B}{Y}\right) &&= \left(1 + C_1\frac{X_H}{1 - X_H}\right)\left(C_2 X_L + X_B\right)X_B. &(5.77) \end{aligned}$$

As for the monolingual case, we assume that due to communication barriers some of the HL couples are unsuccessful with probability $1 - p$. They split up again and form new couples. Analogously to the monolingual case, the distribution of family types is given by

$$\Psi_{HH} = X_H^2 + C_1 X_H(1 - X_H) + (1 - C_1)(1 - D)X_H X_L \tag{5.78}$$

$$\Psi_{HL} = 2D(1 - C_1)X_H X_L \tag{5.79}$$

$$\Psi_{HB} = 2(1 - C_1)X_H X_B \tag{5.80}$$

$$\Psi_{LL} = \left(1 + C_1 \frac{X_H}{1 - X_H}\right) \big(X_L + C_2 X_B\big) X_L \tag{5.81}$$

$$\Psi_{LB} = 2\left(1 + C_1 \frac{X_H}{1 - X_H}\right) (1 - C_2)X_L X_B + (1 - C_1)(1 - D)X_H X_L \tag{5.82}$$

$$\Psi_{BB} = \left(1 + C_1 \frac{X_H}{1 - X_H}\right) \big(C_2 X_L + X_B\big) X_B. \tag{5.83}$$

In the above formulas, D is the same as in the monolingual case, i.e. $D = 1 - (1-p)(1+C_1)/(2-(1-p)(1-C_1)$.

5.2.3 Language transmission

Within families, languages are transmitted from parents to children. By $q_R(F)$ we denote the fraction of F-type families who decide to bring up their children with repertoire R. As in Templin (2018), we make three assumptions:

A1: All children in one family have same language repertoire.

A2: Both parents shall be able to communicate with their children, i.e. $q_L(HH) \equiv q_L(HB) \equiv 0$ and $q_H(LL) \equiv q_H(LB) \equiv 0$.

A3: Only those languages spoken by the parents can be transmitted, i.e. $q_H(HH) \equiv q_L(LL) = 1$.

Since $\sum_R q_R(F) = 1$, we have $q_B(HB) = 1 - q_H(HB)$ and $q_B(LB) = 1 - q_L(LB)$. Conceptualizing individual families as utility maximizing actors, we derive the following properties of the fractions $q_R(F)$, cf. Section 3,

P1: The higher the number of l-monoglots, the higher the incentive to transmit l.

- P1a: If X_H increases, then q_H and q_B do not decrease and q_L does not increase.
- P1b: If X_L increases, then q_L and q_B do not decrease and q_H does not increase.

P2: The higher the status of a language, the higher the incentive to transmit it.

- P2a: If S increases, then $q_H(HB)$ and $q_H(BB)$ do not increase.

P2b: If S increases, then $q_L(LB)$ and $q_L(BB)$ do not decrease.

P3: From properties P1 and P2 it can be deduced that $q_H(HB)$ is independent of X_H and that $q_L(LB)$ is independent of X_H.

Moreover, we assume that not all families are successful in transmitting the repertoire they want to transmit. Linguistic concentration is seen as the main obstacle here. As in Templin (2018), we therefore assume that the fraction of F-type families who successfully transmit repertoire R to their children is given by $Q_R(F)$, where $Q_H(HH) = q_H(HH) \equiv 1$, $Q_L(LL) = q_L(LL) \equiv 1$ and

$$Q_H(HB; C, S, X) = (1 - C_1) \cdot q_H(HB; S, X) + C_1/2 \tag{5.84}$$
$$Q_H(BB; C, S, X) = (1 - C_1) \cdot q_H(BB; S, X) \tag{5.85}$$
$$Q_L(LB; C, S, X) = (1 - C_1) \cdot q_L(LB; S, X) + C_1 \tag{5.86}$$
$$Q_L(BB; C, S, X) = (1 - C_1) \cdot q_L(BB; S, X) + C_1. \tag{5.87}$$

In Templin (2018), we suggested a specific functional form for the fractions $q_R(F)$ that satisfies all the conditions above. For parameters $0 \leq \gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3 \leq 1$, satisfying $0 < \gamma_2 < \delta_3$ and $\delta_1 + \delta_2 < \gamma_1$, these functional expressions are

$$\begin{aligned}
q_H(HB; S, X) &:= \max\{0, \gamma_1(1 - S) - \gamma_2 S X_L\} \\
q_H(BB; S, X) &:= \max\{0, \delta_1(1 - S) + \delta_2(1 - S)X_H - \delta_3 S X_L\} \\
q_L(LB; S, X) &:= \max\{0, \gamma_1 S - \gamma_2(1 - S)X_H\} \\
q_L(BB; S, X) &:= \max\{0, \delta_1 S + \delta_2 S X_L - \delta_3(1 - S)X_H\}.
\end{aligned}$$

In case of only one minority language, this specific functional form works perfectly fine. Extending this version to the case of multiple minorities, which is done later on, we run into problems with $q_{LR}(B_i B_j)$, where B_i and B_j are bilinguals speaking different minority languages (L_i and L_j). Therefore, we present a second version of functional expressions for $q_{LR}(F; S, X)$.

For $x, s \in [0, 1]$, let $h(x, s)$ be a function with values in $[0, 1]$, non-decreasing in x and s and with $h(0, 0) = 0$ and $h(1, 1) = 1$. For h one could e.g. think of $h(x, s) = x \cdot s$, $h(x; s) = (x + s)/2$ or $h(x, s) = (x + s)^2/4$. Given h and non-negative parameters $0 \leq \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2$, we define

$$q_H(HB; S, X) = \varepsilon_1\big(1 - h(X_L, S(L))\big) + \varepsilon_2 S(H) \tag{5.88}$$
$$q_H(BB; S, X) = \zeta_1\big(1 - h(X_L, S(L))\big) + \zeta_2 h(X_H, S(H)) \tag{5.89}$$
$$q_L(LB; S, X) = \varepsilon_1\big(1 - h(X_H, S(H))\big) + \varepsilon_2 S(L) \tag{5.90}$$
$$q_L(BB; S, X) = \zeta_1\big(1 - h(X_H, S(H))\big) + \zeta_2 h(X_L, S(L)) \tag{5.91}$$

with $\varepsilon_1 + \varepsilon_2 = 1$, $\zeta_2 \leq \zeta_1$ and $\zeta_1 + \zeta_2 \leq 1$.

5.2.4 Language dynamics model

In this subsection, we put everything together and present the 1G-1L language dynamics model. Time is measured in years and denoted by t. The number of individuals with repertoire R at time t is denoted by $N_R(t)$, while the fraction is $X_R(t)$. For the dynamic model, we assume that linguistic concentration parameters C_1, C_2 and the status parameters $S(L), S(H)$ are constant.

5.2.4.1 Basic model with language transmission

For the basic model, we only consider population dynamics, family formation and language transmission. The language dynamics can be described by three differential equations

$$\dot{N}_R(t) = -\mu N_R(t) \; + \; \lambda N(t) \sum_F Q_R(F; X(t)) \Psi(F; C, X(t)), \tag{5.92}$$

$R = H, L, B$. The first summand represents the number of people with language repertoire R dying at time t. The second summand represents all the children raised with language repertoire R at time t.

5.2.4.2 Basic model with language learning

Next, we add language acquisition and adult language learning to the model. Recall, language education is captured by the parameters s_{R_1,R_2}, see Section 5.2.1. To improve readability, we introduce the functions

$$f_R(X) := \sum_F Q_R(F; X) \Psi(F; C, X) \tag{5.93}$$

and

$$g_R(X) := (1 - s_{R,B}) f_R(X) + s_{B,R} \cdot f_B(X), \tag{5.94}$$

$R = H, L$. These functions are related to children that leave school with monolingual language repertoire L or H. Moreover, we take into account adult language learning. By $a_{R,B}$, $R = H, L$, we denote the annual rate at which R-monolinguals become bilingual. $a_{R,B}$ is assumed to depend on the linguistic composition, linguistic concentration and relative status of both languages. Hence, $a_{R,B} = a_{R,B}(S(L), C; X)$. In Templin (2018) we offer a functional form for the $a_{R,B}$:

$$a_{L,B}(S(L), C; X) = (1 - C_1) \max\{0, \theta S(H) X_H - \phi(1 - v_H)\}, \tag{5.95}$$

$$a_{H,B}(S(L), C; X) = (1 - C_1) \max\{0, \theta S(L) X_L - \phi(1 - v_L), \tag{5.96}$$

where ϕ, θ are parameters between 0 and 1, and $S(H) = 1 - S(L)$. Writing $a_{R,B}(t)$ for $a_{R,B}(S(L), C; X(t))$, the dynamics of N_H and N_L are described by

$$\dot{N}_R(t) = -\big[\mu + (1 - \mu) a_{R,B}(t)\big] N_R(t) \; + \; \lambda N(t) g_R(X(t)). \tag{5.97}$$

5.2.4.3 Final model with migration

Last, we add migration. The absolute number of people equipped with language repertoire R migrating at time t is denoted by $M_R(t)$. Including migration, the overall population size N changes according to $\dot{N}(t) = (\lambda - \mu)N(t) + M(t)$. The final 1G-1L language competition model that includes transmission, learning and migration is described by the two differential equations, $R = H, L$,

$$\dot{N}_R(t) = -\big[\mu + (1-\mu)a_{R,B}\big]N_R(t) \;+\; \lambda N(t)g_R(X(t)) \;+\; M_R(t). \tag{5.98}$$

Applying the quotient rule to $X_R = N_R/N$, we also obtain

$$\begin{aligned}\dot{X}_R(t) = &-\big[\mu + (1-\mu)a_{R,B}\big]X_R(t) \;+\; \lambda g_R(X(t)) \\ &+ M_R(t)/N(t) - \left(\lambda - \mu + \frac{M(t)}{N(t)}\right).\end{aligned} \tag{5.99}$$

5.2.4.4 Special case: constant migration flow

As a special case, we consider a constant migration flow, that is $M_R(t) \equiv M_R$. For $\lambda < \mu$ the population size N converges to $N(\infty) = M/(\mu - \lambda)$. Plugging this into (5.99), we obtain at the steady state population size

$$\dot{X}_R(t) = -\big[\mu + (1-\mu)a_{R,B}\big]X_R(t) \;+\; \lambda g_R(X(t)) \;+\; (\mu - \lambda)M_R(t)/M(t). \tag{5.100}$$

If $\lambda \geq \mu$ and $M > 0$, then the population size diverges ($N(t) \to \infty$). In this case we get in the limit that

$$\dot{X}_R(t) = -\big[\lambda + (1-\mu)a_{R,B}\big]X_R(t) \;+\; \lambda g_R(X(t)). \tag{5.101}$$

5.2.5 Comparison to earlier models

To a large extend, the 1G-1L model presented in this section was developed in Templin (2018). The two novelties in the above presentation concern family formation and language transmission. For family formation, we added the possibility that L-monolinguals are spatially concentrated with respect to bilinguals. In contrast, in Templin (2018) we assumed $C_2 = 0$. Moreover, in Templin (2018) we assumed that HL-couples are never successful in becoming families, i.e. $p = 0$. Here, HL families can form but with a lower success probability p than all other couple types. For language transmission, we offered here an alternative functional form for the $q_R(F)$ that is suitable for a model with multiple minority languages. All other formulas can already be found in Templin (2018) and are hence presented only very briefly here.

5.3 Models with multiple age groups (3G/10G-1L)

In the 1G-1L model, we do not differentiate between individuals who share the same language repertoire. When deriving the distribution of family types, for example, we consider the total number of mono- and bilinguals. But are all individuals within the population under consideration actually equally relevant for the formation of new families and childbearing? If we ask this question with respect to the age of individuals, the answer is clearly no. Predominantly, young and medium aged adults are the most relevant part of the population for family formation in the above sense. Hence, introducing an age dimension to the 1G-1L model would make the model more sensible to real life dynamics. Including an age dimension could also bring about other advantages. One such advantage is that we can explicitly consider children. In the 1G-1L model we only consider adults. Adults have children, who are equipped with a certain language repertoire by their parents and who expand their repertoire in school. After finishing school, they become part of the adult population. In contrast to reality, though, this does not take 15 or 20 years, but happens in the model – at least implicitly – within a single year. Therefore, changes in the linguistic environment can develop faster (or slower) in the model projections compared to real changes. Another aspect not taken into account by the 1G-1L model is that not all individuals are equally relevant for language learning and language transmission decisions. When deciding on whether to transmit or to learn a language, families and individuals consider its communicative and cultural value as well as learning costs. For example, the benefit of being able to communicate with other people might not be same for all people in the population.[56]

In this section, we present two extensions of the basic 1G-1L model that have an age dimension. We propose a model with three age groups (3G-1L) and a model with 10 age groups (10G-1L). In the 3G-1L model we think of generations (children, younger adult and older adults), while in the 10G-1L model we rather consider 10-year cohorts. In the following, both models are outlined and steady state dynamics are analyzed. Numerical comparisons to the 1G-1L model are presented in Section 5.5.

5.3.1 Model with three generations (3G-1L)

For the three-generation extension of the basic one-generation model we subdivide the population into three age groups. We consider *children* (aged 19 and below), *younger adults* (between 20 and 49 years of age) and *older adults* (50 years of age and above). As in the 1G model, each age group is subdivided along the three language repertoires. For repertoires $R = H, L, B$ we introduce the following notation:

[56] Kennedy & King (2005) provide a simple but illustrating example for that. They develop a model with three overlapping generations and a voting mechanism. They assume that the government collects a lump-sum tax and spends the revenue on language education programs, and that adults vote for the size of the public language programs. From a purely communicative perspective, older adults are not relevant for the voting decision of young adults. In their model, people only live for three generations, so when children of young adults finally reach adulthood themselves, the former old generation has died already. Since languages not only have communicative but also identity value, in our model the repertoires of older adults are taken into account when parents make decisions concerning language transmission.

- n_R: number of children with repertoire R,
- $\mathcal{N}_R$: number of younger adults with repertoire R,
- $\mathfrak{N}_R$: number of older adults with repertoire R.

By $n = n_H + n_L + n_B$ we denote the total number of children. Analogously, $\mathcal{N}$ denotes the total number of younger adults, and $\mathfrak{N}$ the total number of older adults. Furthermore, we introduce the fractions

- $x_R := n_R/n$: fraction of children having language repertoire R,
- $\mathcal{X}_R := \mathcal{N}_R/\mathcal{N}$: fraction of younger adults having language repertoire R,
- $\mathfrak{X}_R := \mathfrak{N}_R/\mathfrak{N}$: fraction of older adults having language repertoire R.

The corresponding vectors are $x = (x_H, x_L)$, $\mathcal{X} = (\mathcal{X}_H, \mathcal{X}_L)$ and $\mathfrak{X} = (\mathfrak{X}_H, \mathfrak{X}_L)$.

5.3.1.1 Family formation and language transmission

As in the original model, it is assumed that individuals form couples and families, have children and transmit languages to the next generation. Which languages are transmitted depends – among other aspects of the environment – on the distribution of family types. The distribution of family types, in turn, depends on the distribution of repertoires throughout the adult population. In the 1G model, we always considered the whole population. But are all people within the population actually relevant for family formation? At a certain age, people might find new partners and form new couples, but do not have new children anymore. If this is not taken into account and if language repertoires are distributed heterogeneously along the age dimension, the model can produce unrealistic projections. Consider, for example, a scenario in which the minority language L is only spoken by older adults, while all younger people are monolingual in H. For family formation, these L speaking older adults are treated the same way as the younger H-monolingual adults. The model therefore assumes that some parents transmit L to their children although parents with young children do not speak L anymore. For the 3G model, we therefore assume that only younger adults (between 20 and 49 years of age) form new families. Children and older adults are not taken into account. Based on (5.78)-(5.83) and setting $p = D = 0$, we have

$$\Psi_{HH} = \mathcal{X}_H^2 + C_1\mathcal{X}_H(1-\mathcal{X}_H) + (1-C_1)\mathcal{X}_H\mathcal{X}_L \tag{5.102}$$
$$\Psi_{HL} = 0 \tag{5.103}$$
$$\Psi_{HB} = 2(1-C_1)\mathcal{X}_H\mathcal{X}_B \tag{5.104}$$
$$\Psi_{LL} = \left(1 + C_1\frac{\mathcal{X}_H}{1-\mathcal{X}_H}\right)\left(\mathcal{X}_L + C_2\mathcal{X}_B\right)\mathcal{X}_L \tag{5.105}$$
$$\Psi_{LB} = 2\left(1 + C_1\frac{\mathcal{X}_H}{1-\mathcal{X}_H}\right)(1-C_2)\mathcal{X}_L\mathcal{X}_B + (1-C_1)\mathcal{X}_H\mathcal{X}_L \tag{5.106}$$
$$\Psi_{BB} = \left(1 + C_1\frac{\mathcal{X}_H}{1-\mathcal{X}_H}\right)\left(C_2\mathcal{X}_L + \mathcal{X}_B\right)\mathcal{X}_B. \tag{5.107}$$

Language transmission within the family is basically modeled as in the 1G case. It is affected by linguistic concentration, the status of the minority language and the linguistic composition of the population. Younger adults might weight the distribution of repertoires among different age groups differently. If they want to choose a repertoire that enables wide communication and future job opportunities for their children, parents might not care so much about the language skills of the old generation. If they want to preserve cultural heritage, the skills of the old generation might be especially relevant. Hence, we introduce weights π_x (children), $\pi_{\mathcal{X}}$ (younger adults), and $\pi_{\mathfrak{X}}$ (older adults). We require $\pi_x + \pi_{\mathcal{X}} + \pi_{\mathfrak{X}} = 1$. Based on these weights, we define the weighted distribution of language repertoires

$$\chi_R(x, \mathcal{X}, \mathfrak{X}) := \pi_x x_R + \pi_{\mathcal{X}} \mathcal{X}_R + \pi_{\mathfrak{X}} \mathfrak{X}_R,$$

$R = H, L, B$. Let $\chi = (\chi_H, \chi_L)$. For example, if only younger adults are considered relevant, then $\pi_{\mathcal{X}} = 1$ and $\chi = \mathcal{X}$. This weighted distribution then determines which languages are transmitted. That means we set

$$Q_R(F) = Q_R(F; C, S, \chi), \tag{5.108}$$

where $Q_R(F; C, S, \cdot)$ is defined by (5.84)-(5.87) and (5.88)-(5.91). So instead of the distribution of repertoires in the whole population X, we work with the weighted distribution χ.

5.3.1.2 Language dynamics

Language dynamics in the 3G model are similar to those in the basic 1G model. One major difference is that only younger adults are assumed to form new families and produce offspring. To capture the effect of family formation and language transmission we define analogously to (5.93),

$$f_R(x, \mathcal{X}, \mathfrak{X}) := \sum_F Q_R\big(F; \chi(x, \mathcal{X}, \mathfrak{X})\big) \cdot \Psi(F; C, \mathcal{X}). \tag{5.109}$$

In the 3G model, the dynamics of the three generations have to be described separately. We assume that individual age is equally distributed in each of the three age groups. That means that there are as many two year old children as there are 18 year old children, and as many 25 year old younger adults as 45 year old younger adults. Moreover, we make the simplifying assumption that death occurs only among older adults. The death rate in this age group is denoted by $\mu_{\mathfrak{X}}$ and is higher than the overall death rate μ in the 1G model. By $\lambda_{\mathcal{X}}$ we denote the birth rate among younger adults, which is higher than the overall birth rate λ in the 1G model. The basic dynamics without schooling, adult language learning and migration are given by the following six differential equations:

$$\dot{n}_R(t) = -\frac{n_R(t)}{20} + \lambda_{\mathcal{X}} \mathcal{N}(t) f_R(x(t), \mathcal{X}(t), \mathfrak{X}(t)) \tag{5.110}$$

$$\dot{\mathcal{N}}_R(t) = -\frac{\mathcal{N}_R(t)}{30} + \frac{n_R(t)}{20}, \tag{5.111}$$

$$\dot{\mathfrak{N}}_R(t) = -\mu_{\mathfrak{X}} \mathfrak{N}_R(t) + \frac{\mathcal{N}_R(t)}{30}, \tag{5.112}$$

$R = H, L$. The numbers 20 and 30 correspond to the age-span of children and younger adults. Next, we add schooling, adult language learning and migration. We assume that only younger adults learn additional languages. As before, the more people speak a language, the more attractive it is to learn this language. For language transmission we already argued that not all individuals in the population are equally relevant for parents' decisions. The same holds true for adult language learning. Hence, instead of X in the 1G model, adults consider the weighted distribution χ for their learning decisions. We hence use

$$a_{L,B}(S(L), C; \chi) = (1 - C_1) \max\{0, \theta S(H)\chi_H - \phi(1 - v_H)\} \tag{5.113}$$
$$a_{H,B}(S(L), C; \chi) = (1 - C_1) \max\{0, \theta S(L)\chi_L - \phi(1 - v_L). \tag{5.114}$$

To shorten the formulas, we define $f_R(t) = f_R(x(t), \mathcal{X}(t), \mathfrak{X}(t))$ and $a_{R,B}(t) = a_{R,B}(S(L), C; \chi(t))$. We denote the number of children migrating to the population in year t by $m(t)$, the number of younger adults by $\mathcal{M}(t)$ and the number of older adults by $\mathfrak{M}(t)$. The language dynamics of the 3G model are described by

$$\dot{n}_R(t) = -\frac{n_R(t)}{20} + \lambda_{\mathcal{X}} \mathcal{N}(t) f_R(t) + m_R(t), \tag{5.115}$$

$$\dot{\mathcal{N}}_R(t) = -\frac{1 + 29a_{R,B}(t)}{30}\mathcal{N}_R(t) + \frac{(1 - s_{R,B})n_R(t) + s_{B,R}n_B(t)}{20} + \mathcal{M}_R(t), \tag{5.116}$$

$$\dot{\mathfrak{N}}_R(t) = -\mu_{\mathfrak{X}}\mathfrak{N}_R(t) + \frac{\mathcal{N}_R(t)}{30} + \mathfrak{M}_R(t), \tag{5.117}$$

$R = H, L$.

5.3.1.3 Special case: constant migration flow

As for the 1G model, we consider the special case of a constant migration flow, i.e. $m(t) \equiv m$, $\mathcal{M}(t) \equiv \mathcal{M}$ and $\mathfrak{M}(t) \equiv \mathfrak{M}$. Consequently, the overall annual number of migrants $M = m + \mathcal{M} + \mathfrak{M}$ is constant as well. We introduce the notation $q := m/M$, $\mathcal{Q} := \mathcal{M}/M$ and $\mathfrak{Q} := \mathfrak{M}/M$. Moreover, we assume that the distribution of the newcomers' language repertoires is the same for all age groups, i.e. $m_R/m = \mathcal{M}_R/\mathcal{M} = \mathfrak{M}_R/\mathfrak{M} =: \mathbb{Q}_R$.

Again, we want to calculate the dynamics of the distribution of repertoires if the population size is at a steady state. We know that the sizes of the age groups evolve according to

$$\dot{n}(t) = -\frac{1}{20}n(t) + \lambda_{\mathcal{X}}\mathcal{N}(t) + m(t), \tag{5.118}$$
$$\dot{\mathcal{N}}(t) = -\frac{1}{30}\mathcal{N}(t) + \frac{1}{20}n(t) + \mathcal{M}(t), \tag{5.119}$$
$$\dot{\mathfrak{N}}(t) = -\mu_{\mathfrak{X}}\mathfrak{N}(t) + \frac{1}{30}\mathcal{N}(t) + \mathfrak{M}(t). \tag{5.120}$$

Lemma 5.3.1.1. *For $\lambda_{\mathcal{X}} < 1/30$ the dynamic system described by* (5.118)-(5.120) *is asymptotically stable. The steady states are given by*

$$n_\infty = 20\left(\lambda_{\mathcal{X}}\frac{1-\mathfrak{Q}}{1/30-\lambda_{\mathcal{X}}}+q\right)M \qquad =: w_\infty M, \tag{5.121}$$

$$\mathcal{N}_\infty = \frac{1-\mathfrak{Q}}{1/30-\lambda_{\mathcal{X}}}M \qquad =: \mathcal{W}_\infty M, \tag{5.122}$$

$$\mathfrak{N}_\infty = \frac{1}{\mu_{\mathfrak{X}}}\left(\frac{1-\mathfrak{Q}}{1-30\lambda_{\mathcal{X}}}+\mathfrak{Q}\right)M \qquad =: \mathfrak{W}_\infty M. \tag{5.123}$$

The proof of this lemma can be found in the Appendix. For $\lambda_{\mathcal{X}} \geq 1/30$, the overall population size grows to infinity over time, as do the absolute sizes of the three age groups. In the next lemma we consider the relation between the sizes of the three age groups.

Lemma 5.3.1.2. *Let $\lambda_{\mathcal{X}} \geq 1/30$. Then,*

$$W_{\mathcal{N}/n} := \lim_{t\to\infty}\frac{\mathcal{N}(t)}{n(t)} = \frac{1}{120\lambda_{\mathcal{X}}}+\sqrt{\frac{1}{(120\lambda_{\mathcal{X}})^2}+\frac{1}{20\lambda_{\mathcal{X}}}}, \tag{5.124}$$

$$W_{n/\mathcal{N}} := \lim_{t\to\infty}\frac{n(t)}{\mathcal{N}(t)} = 1/L_{\mathcal{N}/n}, \tag{5.125}$$

$$W_{\mathcal{N}/\mathfrak{N}} := \lim_{t\to\infty}\frac{\mathcal{N}(t)}{\mathfrak{N}(t)} = 30\mu_{\mathfrak{X}}-1+\frac{3}{2}L_{n,\mathcal{N}}. \tag{5.126}$$

This lemma is proven in the Appendix.

With the help of the last two lemmas, we now look at the distribution of language skills when the population sizes are at their steady states or steady state fractions. Applying the quotient rule, we get

$$\dot{x}_R(t) = \lambda_{\mathcal{X}}\big(f_R(t)-x_R(t)\big)\frac{\mathcal{N}(t)}{n(t)}+\big(\mathbb{Q}_R-x_R(t)\big)\frac{qM(t)}{n(t)} \tag{5.127}$$

$$\begin{aligned}\dot{\mathcal{X}}_R(t) = &-\frac{29}{30}a_{R,B}(t)\mathcal{X}_R(t)+\frac{1}{20}\big[(1-s_{R,B})x_R(t)+s_{B,R}x_B(t)-\mathcal{X}_R(t)\big]\frac{n(t)}{\mathcal{N}(t)}\\ &+\big(\mathbb{Q}_R-\mathcal{X}_R(t)\big)\frac{\mathcal{Q}M(t)}{\mathcal{N}(t)}\end{aligned} \tag{5.128}$$

$$\dot{\mathfrak{X}}_R(t) = \frac{1}{30}\big(\mathcal{X}_R(t)-\mathfrak{X}_R(t)\big)\frac{\mathcal{N}(t)}{\mathfrak{N}(t)}+\big(\mathbb{Q}_R-\mathfrak{X}_R(t)\big)\frac{\mathfrak{Q}M(t)}{\mathfrak{N}(t)} \tag{5.129}$$

In the case of $\lambda_{\mathcal{X}} < 1/30$, the steady states of the system $(n(t), \mathcal{N}(t), \mathfrak{N}(t))$ are given by (5.121)-(5.123). Plugging these in, we get in the steady state that

$$\dot{x}_R(t) = \lambda_{\mathcal{X}}\big(f_R(t) - x_R(t)\big)\frac{\mathcal{W}_\infty}{w_\infty} + \big(\mathbb{Q}_R - x_R(t)\big)\frac{q}{w_\infty}, \tag{5.130}$$

$$\begin{aligned}\dot{\mathcal{X}}_R(t) = &-\frac{29}{30}a_{R,B}(t)\mathcal{X}_R(t) + \frac{1}{20}\big[(1-s_{R,B})x_R(t) + s_{B,R}x_B(t) - \mathcal{X}_R(t))\big]\frac{w_\infty}{\mathcal{W}_\infty}\\ &+ \big(\mathbb{Q}_R - \mathcal{X}_R(t)\big)\frac{\mathcal{Q}}{\mathcal{W}_\infty},\end{aligned} \tag{5.131}$$

$$\dot{\mathfrak{X}}_R(t) = \frac{1}{30}\big(\mathcal{X}_R(t) - \mathfrak{X}_R(t)\big)\frac{\mathcal{W}_\infty}{\mathfrak{W}_\infty} + \big(\mathbb{Q}_R - \mathfrak{X}_R(t)\big)\frac{\mathfrak{Q}}{\mathfrak{W}_\infty}. \tag{5.132}$$

Using Lemma 5.3.1.2, we have for $\lambda_{\mathcal{X}} \geq 1/30$ that

$$\dot{x}_R(t) = \lambda_{\mathcal{X}}\big(f_R(t) - x_R(t)\big)W_{\mathcal{N}/n}, \tag{5.133}$$

$$\begin{aligned}\dot{\mathcal{X}}_R(t) = &-\frac{29}{30}a_{R,B}(t)\mathcal{X}_R(t)\\ &+ \frac{1}{20}\big[(1-s_{R,B})x_R(t) + s_{B,R}x_B(t) - \mathcal{X}_R(t))\big]W_{n/\mathcal{N}},\end{aligned} \tag{5.134}$$

$$\dot{\mathfrak{X}}_R(t) = \frac{1}{30}\big(\mathcal{X}_R(t) - \mathfrak{X}_R(t)\big)W_{\mathcal{N}/\mathfrak{N}}. \tag{5.135}$$

5.3.2 Model with 10 age groups

For the 10G model, we distinguish not between three generations, but between 10 age groups. For $R = H, L, B$ and $i = 1, ..., 10$, we denote by $N_{i,R}$ the number of people with language repertoire R aged between $10(i-1)$ years and $10i - 1$ years. For example, $N_{3,H}$ is the number of H-monolinguals between 20 and 29. Furthermore, for $i = 1, ..., 10$, N_i denotes the number of all individuals between $10(i-1)$ and $10i - 1$ years of age. The relative sizes of the language repertoire and age groups are denoted by $X_{i,R} := N_{i,R}/N_i$. Here, we also use the notation introduced for the 3G model:

$$n_R := N_{1,R} + N_{2,R}, \tag{5.136}$$
$$\mathcal{N}_R := N_{3,R} + N_{4,R} + N_{5,R}, \tag{5.137}$$
$$\mathfrak{N}_R := N_{6,R} + N_{7,R} + N_{8,R} + N_{9,R} + N_{10,R}, \tag{5.138}$$

and

$$x_R := \frac{\sum_{i=1}^{2} N_{i,R}}{\sum_{i=1}^{2} N_i}, \tag{5.139}$$

$$\mathcal{X}_R := \frac{\sum_{i=3}^{5} N_{i,R}}{\sum_{i=3}^{5} N_i}, \tag{5.140}$$

$$\mathfrak{X}_R := \frac{\sum_{i=6}^{10} N_{i,R}}{\sum_{i=6}^{10} N_i}. \tag{5.141}$$

As before, $x = (x_H, x_L)$, $\mathcal{X} = (\mathcal{X}_H, \mathcal{X}_L)$ and $\mathfrak{X} = (\mathfrak{X}_H, \mathfrak{X}_L)$. Concerning language learning and family formation, we make the same assumptions as in the 3G model: only younger adults, i.e. individuals between 20 and 49 (age groups 3-5), learn additional languages outside of formal education and form new families. Moreover, we again assume that ages are equally distributed in each age group, i.e. there are as many 25 year old younger adults as there are 29 year old younger adults. Last, we assume that only older adults (age groups 6-10) die.

5.3.2.1 Family formation and language transmission

Family formation and language transmission are modeled as in the 3G case. Using $\mathcal{N}$ as defined in (5.137), the fractions of families of type F, $\Psi_F = \Psi_F(F; C, \mathcal{X})$, are determined by (5.102)-(5.107). Using the weighted distribution

$$\chi_R(x, \mathcal{X}, \mathfrak{X}) := \pi_x x_R + \pi_{\mathcal{X}} \mathcal{X}_R + \pi_{\mathfrak{X}} \mathfrak{X}_R$$

and $\chi = (\chi_H, \chi_L)$, we assume that the fraction of F-type families bringing up their children with repertoire R is $Q_R(F; C, S, \chi)$, cf. (5.108). We hence use the same short notation

$$\begin{aligned} f_R(t) &= f_R\big(C, x(t), \mathcal{X}(t), \mathfrak{X}(t)\big) \\ &= \sum_F Q_R\big(F; \chi(x(t), \mathcal{X}(t), \mathfrak{X}(t))\big) \cdot \Psi\big(F; C, \mathcal{X}(t)\big), \\ a_{R,B}(t) &= a_{r_B}\big(S(L), C; \chi(x(t), \mathcal{X}(t), \mathfrak{X}(t))\big). \end{aligned}$$

5.3.2.2 Language dynamics

Despite some differences, the dynamics of the 10G model are comparable to those of the 3G model. In the 10G model, the repertoires of the two youngest age groups are affected by schooling. To differentiate between primary and secondary education, we use two sets of schooling parameters: $s_{1,R,B}$ and $s_{2,R,B}$. The parameter $s_{1,R,B}$ denotes the fraction of children raised with language repertoire R, $R = H, L$, who become bilingual due to primary education (to be more precise: children raised with R that are bilingual at the age of 10). The parameter $s_{2,R,B}$ corresponds to secondary education: it denotes the fraction of children that are monolingual at the age of 10 and acquire an additional language until the age of 20. As in the 3G model, we make the simplifying assumption that the death rate $\mu_{\mathfrak{X}}$ is the same for all older adults. Moreover, we assume that people are not getting older than 99 years. We denote the number of people migrating with repertoire R to age group i at time t by $M_{i,R}(t)$. For $i = 4, 5$ and $j = 7, ..., 10$, the dynamics are given by

$$\dot{N}_{1,R}(t) = -\frac{1}{10}N_{1,R}(t) + \lambda_{\mathcal{X}}\mathcal{N}(t)f_R(t) + M_{1,R}(t), \tag{5.142}$$

$$\dot{N}_{2,R}(t) = \big(-N_{2,R}(t) + (1-s_{1,R,B})N_{1,R}(t) + s_{1,B,R}N_{1,B}(t)\big)/10 + M_{2,R}(t), \tag{5.143}$$

$$\begin{aligned}\dot{N}_{3,R}(t) = &-\big(1+9a_{R,B}(\mathcal{X}(t))\big)N_{3,R}(t)/10 + (1-s_{2,R,B})N_{2,R}(t)/10 \\ &+ s_{2,B,R}N_{2,B}(t)/10 + M_{3,R}(t),\end{aligned} \tag{5.144}$$

$$\begin{aligned}\dot{N}_{i,R}(t) = &\Bigg(-\big(1+9a_{R,B}(\mathcal{X}(t))\big)N_{i,R}(t) + \big(1-a_{R,B}(\mathcal{X}(t)\big)N_{i-1,R}(t)\Bigg)/10 \\ &+ M_{i,R}(t),\end{aligned} \tag{5.145}$$

$$\dot{N}_{6,R}(t) = \Bigg(-(1+9\mu_{\mathfrak{X}})N_{6,R}(t) + \big(1-a_{R,B}(\mathcal{X}(t)\big)N_{5,R}(t)\Bigg)/10 + M_{6,R}(t), \tag{5.146}$$

$$\dot{N}_{j,R}(t) = \Bigg(-(1+9\mu_{\mathfrak{X}})N_{j,R}(t) + (1-\mu_{\mathfrak{X}})N_{j-1,R}(t)\Bigg)/10 + M_{j,R}(t). \tag{5.147}$$

5.3.2.3 Special case: constant migration flow

Again, we take a closer look at the language dynamics in case of a constant migration flow. So $M_i(t) \equiv M_i$ for all t. By q_i we denote the fraction M_i/M. As before, we assume that $M_{i,R}/M_i =: \mathbb{Q}_R$ is the same for all i. The overall sizes of all ten age groups evolve according to

$$\dot{N}_1 = -N_1/10 + \lambda_{\mathcal{X}}\mathcal{N} + M_1, \tag{5.148}$$

$$\dot{N}_i = -N_i/10 + N_{i-1}/10 + M_i, \tag{5.149}$$

$$\dot{N}_6 = -(\mu_{\mathfrak{X}} + (1-\mu_{\mathfrak{X}})/10)N_6 + N_5/10 + M_6, \tag{5.150}$$

$$\dot{N}_j = -(\mu_{\mathfrak{X}} + (1-\mu_{\mathfrak{X}})/10)N_j + (1-\mu_{\mathfrak{X}})N_{j-1}/10 + M_j, \tag{5.151}$$

where $i = 2,3,4,5$ and $j = 7,8,9,10$. Note, for $i \geq 2$ the above differential equations are of the form

$$\dot{N}_i = -A_iN_i + B_iN_{i-1} + M_i,$$

for constant coefficients A_i, B_i.

As for the 3G model, we want to analyze steady states of the system. We have to consider the convergent case $\lambda_{\mathcal{X}} < 1/30$ and the divergent case $\lambda_{\mathcal{X}} \geq 1/30$ separately. The two following lemmas hold true.

Lemma 5.3.2.1. *For $\lambda_{\mathcal{X}} < 1/30$, the dynamic system described by* (5.148)-(5.151) *is dynamically stable. To express the steady states of the system, we define $D_1 = 1$, $E_1 = 0$ and for $i = 2, ..., 10$,*

$$D_i := \prod_{j=2}^{i}\frac{B_j}{A_j}, \qquad E_i := \frac{q_i}{A_i} + \sum_{k=2}^{i-1}\left(\prod_{j=k+1}^{i}\frac{B_j}{A_j}\right)\frac{q_k}{A_k}.$$

Moreover, we define $\mathcal{D} = D_3 + D_4 + D_5$ *and* $\mathcal{E} = E_3 + E_4 + E_5$*. The steady states can be expressed as,* $i = 1, ..., 10$*,*

$$N_{i,\infty} = \left[D_i \frac{10}{1 - 10\lambda_{\mathcal{X}}\mathcal{D}} \big(\lambda_{\mathcal{X}}\mathcal{E} + q_1\big) + E_i \right] M =: F_{i,\infty} M \tag{5.152}$$

A proof is provided in the Appendix. Consequently, the number of younger adults in the steady state is

$$\mathcal{N}_\infty = M \sum_{i=3}^{5} F_{i,\infty} =: \mathcal{F}_\infty M. \tag{5.153}$$

Next, we derive the differential equations for the distribution of the language repertoires for each age group in the steady state population. To do so, we once more apply the quotient rule, i.e.

$$\dot{X}_{i,R}(t) = \frac{d}{dt}\left(\frac{N_{i,R}(t)}{N_i(t)}\right) = \frac{\dot{N}_{i,R}(t)}{N_i(t)} - X_{i,R}(t).\frac{\dot{N}_i(t)}{N_i(t)}. \tag{5.154}$$

In detail, this yields for $i = 4, 5$ and $j = 7, 8, 9, 10$,

$$\dot{X}_{1,R}(t) = \lambda_{\mathcal{X}}\big(f_R(t) - X_{1,R}(t)\big)\frac{\mathcal{N}(t)}{N_1(t)} + q_1\big(\mathbb{Q}_R - X_{1,R}(t)\big)\frac{M}{N_1(t)}, \tag{5.155}$$

$$\begin{aligned}\dot{X}_{2,R}(t) = {} & \frac{1}{10}\Big((1 - s_{1,R,B})X_{1,R}(t) - X_{2,R}(t)\Big)\frac{N_1(t)}{N_2(t)} \\ & + q_2\big(\mathbb{Q}_R - X_{2,R}(t)\big)\frac{M}{N_2(t)},\end{aligned} \tag{5.156}$$

$$\begin{aligned}\dot{X}_{3,R}(t) = {} & -\frac{9a_{R,B}(\mathcal{X}(t))}{10}X_{3,R}(t) + \frac{1}{10}\Big((1 - s_{2,R,B})X_{2,R}(t) - X_{3,R}(t)\Big)\frac{N_2(t)}{N_3(t)} \\ & + q_3\big(\mathbb{Q}_R - X_{3,R}(t)\big)\frac{M}{N_3(t)},\end{aligned} \tag{5.157}$$

$$\begin{aligned}\dot{X}_{i,R}(t) = {} & -\frac{9a_{R,B}(\mathcal{X}(t))}{10}X_{i,R}(t) \\ & + \frac{1}{10}\Big((1 - a_{R,B}(\mathcal{X}(t)))X_{i-1,R}(t) - X_{i,R}(t)\Big)\frac{N_{i-1}(t)}{N_i(t)} \\ & + q_i\big(\mathbb{Q}_R - X_{i,R}(t)\big)\frac{M}{N_i(t)},\end{aligned} \tag{5.158}$$

$$\begin{aligned}\dot{X}_{6,R}(t) = {} & \frac{1}{10}\Big(\big(1 - a_{R,B}(\mathcal{X}(t))\big)X_{5,R}(t) - X_{6,R}(t)\Big)\frac{N_5(t)}{N_6(t)} \\ & + q_6\big(\mathbb{Q}_R - X_{6,R}(t)\big)\frac{M}{N_6(t)},\end{aligned} \tag{5.159}$$

$$\dot{X}_{j,R}(t) = \frac{1 - \mu_{\mathfrak{X}}}{10}\Big(X_{j-1,R}(t) - X_{j,R}(t)\Big)\frac{N_{j-1}(t)}{N_j(t)} + q_j\big(\mathbb{Q}_R - X_{j,R}(t)\big)\frac{M}{N_j(t)}. \tag{5.160}$$

In the case of $\lambda_{\mathcal{X}} < 1/30$, we can plug in $N_{i,\infty} = F_{i,\infty}M$ for $N_i(t)$, $i = 1, ..., 10$, cf. (5.152) and (5.153) to obtain the dynamics of the linguistic composition in the steady state. For $\lambda_{\mathcal{X}} \geq 1/30$, all N_i tend to infinity as $t \to \infty$. Hence, the last term $q_k(\mathbb{Q}_R - X_{k,R}(t))M/N_k(t)$ vanishes in the limit. To handle the first term(s) in equations (5.155)-(5.160), we have to investigate the limits $N_k(t)/N_{k+1}(t)$ as $t \to \infty$.

Lemma 5.3.2.2. *Consider N_k, $k = 1, ..., 10$, as defined by* (5.148)-(5.151). *Let $\lambda_{\mathcal{X}} \geq 1/30$ and let W_∞ be the unique positive real solution of the polynomial equation $0 = W_\infty^5 - 10\lambda_{\mathcal{X}}(W_\infty^2 + W_\infty + 1)$. Then, for $i = 1, 2, 3, 4$ and $j = 6, 7, 8, 9$, we have*

$$\lim_{t\to\infty} \frac{N_i(t)}{N_{i+1}(t)} = W_\infty, \tag{5.161}$$

$$\lim_{t\to\infty} \frac{N_5(t)}{N_6(t)} = W_\infty + 9\mu_{\mathfrak{X}}, \tag{5.162}$$

$$\lim_{t\to\infty} \frac{N_j(t)}{N_{j+1}(t)} = \frac{W_\infty + 9\mu_{\mathfrak{X}}}{1 - \mu_{\mathfrak{X}}}. \tag{5.163}$$

As for $\lambda_{\mathcal{X}} < 1/30$, we can now plug in these limits and obtain differential equations for the evolution of the relative sizes of the language repertoire groups in the limit. Note, for $\lambda_{\mathcal{X}} = 1/30$ we have $W_\infty = 1$ and that W_∞ increases with $\lambda_{\mathcal{X}}$.

5.4 A model with several minority languages (1G-nL)

In this section, we present a version of the 1G model with multiple minorities. We consider a majority language H (the official local language) and n minority languages, denoted by L_i, $i = 1, .., n$. We denote the set of indexes $\{i | i = 1, ..., n\}$ by I. We assume that every individual is either monolingual in H, monolingual in one of the minority languages L_i, or speaks H and additionally a minority language. A bilingual speaker of H and L_i is denoted by B_i. For the model, we neglect the case that individuals could speak multiple minority languages. By $\mathfrak{R}$ we denote the set of all “relevant” language repertoires, i.e.

$$\mathfrak{R} = \{H, L_1, ..., L_n, B_1, ..., B_n\}.$$

All together we have $\#\mathfrak{R} = 1 + 2n$ relevant language repertoires. Next we introduce some notation. Let $R \in \mathfrak{R}$. We define

- N: overall population size,
- N_R: Number of people with repertoire R,
- $N_L := N_{L_1} + ... + N_{L_n}$: Number of monolingual speakers of minority languages,
- $N_B := N_{B_1} + ... + N_{B_n}$: Number of bilinguals.

We also consider the fractions $X_R = N_R/N$, $X_L = N_L/N$ and $X_B = N_B/N$. Moreover, we introduce the column vector
$X := (X_H, X_{L_1}, ..., X_{L_n}, X_{B_1}, ..., X_{B_n})'$.

Status parameters
For the models with two languages, we only had to consider the status of the majority language and the status of the single minority language. Here, we have to take into account the status of all the minority languages. Let $\overline{S}_H$ be the status of the majority language and $\overline{S}_{L_i}$ the status of minority language L_i. We consider the relative status of languages l, $l = H, L_1, ..., L_n$:

$$S_l = \frac{\overline{S}_l}{S_H + S_{L_1} + ... + S_{L_n}}. \tag{5.164}$$

By construction,

$$S_H + \sum_{i=1}^{n} S_{L_i} = 1.$$

Let S_L denote the vector $(S_{L_1}, ..., S_{L_n})'$ and S denote the vector $(S_H, S_{L_1}, ..., S_{L_n})'$.

Concentration parameters
For the nL model, we work with n pairs of concentration parameters $C_{1,i}, C_{2,i}$, $i = 1, ..., n$. The parameter $C_{1,i}$ measures the concentration of speakers of minority language L_i (with respect to H monolinguals), and the parameter $C_{2,i}$ measures the concentration of L_i monolinguals with respect to bilinguals speaking H and L_i. As before, let the territory be divided in K regions. Analogously to the 1L model, we consider

$$C_{1,i} = \frac{1}{2} \sum_{k=1}^{K} \left| \frac{N_{L_i,k} + N_{B_i,k}}{N_{L_i} + N_{B_i}} - \frac{N_{H,k}}{N_H} \right|, \tag{5.165}$$

$$C_{2,i} = \frac{1}{2} \sum_{k=1}^{K} \left| \frac{N_{L_i,k}}{N_{L_i}} - \frac{N_{B_i,k}}{N_{B_i}} \right|, \tag{5.166}$$

and the matrix

$$C = \begin{pmatrix} C_{1,1} & ... & C_{1,n} \\ C_{2,1} & ... & C_{2,n} \end{pmatrix}.$$

5.4.1 Family formation

Family formation and language transmission are modeled comparably to the case of only two languages (1G-1L). Couples and families consist of two individuals. Considering all possible combinations of repertoires, we define different sets of family types:

$$\mathbb{F}_{HL} := \{HL_i \,|\, i \in I\}, \tag{5.167}$$
$$\mathbb{F}_{HB} := \{HB_i \,|\, i \in I\}, \tag{5.168}$$
$$\mathbb{F}_{LL} := \{L_iL_j \,|\, i,j \in I\}, \tag{5.169}$$
$$\mathbb{F}_{BB} := \{B_iB_j \,|\, i,j \in I\}, \tag{5.170}$$
$$\mathbb{F}_{LB} := \{L_iB_j \,|\, i,j \in I\}. \tag{5.171}$$

The set of all possible family types $\mathbb{F}$ is the given by $\mathbb{F} = \{HH\} \cup \mathbb{F}_{HL} \cup \mathbb{F}_{HB} \cup \mathbb{F}_{BB} \cup \mathbb{F}_{LL} \cup \mathbb{F}_{LB}$. Hence, the number of family types is given by

$$\#\mathbb{F} = 1 + n + n + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + n^2 = 1 + 3n + 2n^2.$$

For $n = 1$ we get $\#\mathbb{F} = 6$ and for $n = 2$ we already have $\#\mathbb{F} = 15$.

Lemma 5.4.1.1. *In the 1G-nL model, couples are distributed as follows:*

$$\psi_{HH} = X_H^2 + X_H \sum_{j=1}^{n} C_{1,j} X_j, \tag{5.172}$$
$$\psi_{HL_i} = (1 - C_{1,i})(1 + X_H)X_{L_i} - X_{L_i} \sum_{j=1}^{n} (1 - C_{1,j})X_j, \tag{5.173}$$
$$\psi_{HB_i} = (1 - C_{1,i})(1 + X_H)X_{B_i} - X_{B_i} \sum_{j=1}^{n} (1 - C_{1,j})X_j, \tag{5.174}$$
$$\psi_{L_iL_j} = (2 - C_{1,i} - C_{1,j})X_{L_i}X_{L_j}, \tag{5.175}$$
$$\psi_{L_iB_j} = (2 - C_{1,i} - C_{1,j})X_{L_i}X_{B_j}, \tag{5.176}$$
$$\psi_{B_iB_j} = (2 - C_{1,i} - C_{1,j})X_{B_i}X_{B_j}, \tag{5.177}$$
$$\psi_{L_iL_i} + \psi_{L_iB_i} + \psi_{B_iB_i} = X_i\big[C_{1,i} + (1 - C_{1,i})X_i\big]. \tag{5.178}$$

Defining $E_i := C_{1,i} + (1 - C_{1,i})X_i$, we further have

$$\psi_{L_iL_i} = E_i(X_{L_i} + C_{2,i}X_{B_i})\frac{X_{L_i}}{X_i}, \tag{5.179}$$
$$\psi_{L_iB_i} = 2E_i(1 - C_{2,i})\frac{X_{L_i}X_{B_i}}{X_i}, \tag{5.180}$$
$$\psi_{B_iB_i} = E_i(X_{B_i} + C_{2,i}X_{L_i})\frac{X_{B_i}}{X_i}. \tag{5.181}$$

A derivation of the couple distribution in the 1G-nL model is provided in the Appendix. For $n = 1$, the above formulas yield the exact same couple distribution we derived for the 1G-1L model. Given this couple type distribution, one can now derive the family type distribution. The procedure is as before. Individuals form couples. Those couples with a common language ($HH, HB_i, L_iL_i, L_iB_i, B_iB_j$) are assumed to be always successful, i.e. become a family. All those couples without a common language split up again ($p = 0$), due to communication barriers. They enter a new round of couple and family formation. This is repeated until all individuals are part of a family. Since formulas are more complex for the nL case than for the 1L case, we do not provide lengthy derivations and formulas for the family distribution Ψ for the nL model. Instead, for an application of the model, the family type distribution is approximated numerically. In Table 9, we provide a numerical example to compare the results of the random family formation process for a 1L and 2L model. For the 1G model, we set $X_H = 0.7$, $X_L = 0.1$ and $X_B = 0.2$. For the 2G model, we consider two examples matching the numbers of the 1G model. As a first example, we consider the symmetric example $X_H = 0.7$, $X_{L_1} = X_{L_2} = 0.05$ and $X_{B_1} = X_{B_2} = 0.1$. As a second example, we consider a case with a larger and a smaller minority language: $X_H = 0.7$, $X_{L_1} = 0.08, X_{L_2} = 0.02$ and $X_{B_1} = 0.16, X_{B_2} = 0.04$. We set for the concentration measures $C_{1,1} = C_{1,2} = 0.5$ and $C_{2,1} = C_{2,2} = 0$. This results in the family type distributions shown in Table 9.

5.4.2 Language transmission

As before, parents can raise their children as monolinguals ($R = H, L_1, ..., L_n$) or as bilinguals ($R = B_1, ..., B_n$). For $F \in \mathbb{F}$, we consider $q_R(F)$, the fraction of F-type families who decide to bring up their children with language repertoire R.

Before introducing functional expressions for the q-functions, we use the utility maximization approach to derive general properties of those functions. As for the 1L case, we make the following assumptions:

A1: All children in one family have same language repertoire.

A2: Both parents shall be able to communicate with their children, i.e. $q_{L_i}(HH) \equiv q_{L_i}(HB_i) \equiv q_{L_i}(HB_j) \equiv q_{L_i}(L_jB_j) \equiv q_{L_i}(B_jB_j) \equiv 0$ and $q_H(L_iL_i) \equiv q_H(L_iB_i) \equiv 0$.

A3: Only those languages spoken by the parents can be transmitted, i.e. $q_H(HH) \equiv q_{L_i}(L_iL_i) \equiv 1$.

Assumptions A2 and A3 yield, $i, j = 1, .., n$, $i \neq j$,

$$\begin{aligned}
1 &= q_H(HB_i) + q_{B_i}(HB_i), \\
1 &= q_{L_i}(L_iB_i) + q_{B_i}(L_iB_i), \\
1 &= q_H(B_iB_i) + q_{L_i}(B_iB_i) + q_{B_i}(B_iB_i), \\
1 &= q_H(B_iB_j) + q_{B_i}(B_iB_j) + q_{B_j}(B_iB_j).
\end{aligned}$$

Using the utility maximization approach, we obtain general properties of the functions $q_R(F)$ depicted in Table 10. A justification for the properties is provided in the Appendix.

1L model		**2L model**		
	Example 1 & 2		Example 1	Example 2
Ψ_{HH}	0.6300	Ψ_{HH}	0.6288	0.6292
Ψ_{HL}	0	Ψ_{HL_1}	0	0
		Ψ_{HL_2}	0	0
		$\sum$	0	0
Ψ_{HB}	0.1400	Ψ_{HB_1}	0.0712	0.1128
		Ψ_{HB_2}	0.0712	0.0289
		$\sum$	0.1424	0.1416
Ψ_{LL}	0.0567	$\Psi_{L_1L_1}$	0.0292	0.0458
		$\Psi_{L_2L_2}$	0.0292	0.0119
		$\Psi_{L_1L_2}$	0	0
		$\sum$	0.0583	0.0577
Ψ_{LB}	0.0867	$\Psi_{L_1B_1}$	0.0417	0.0684
		$\Psi_{L_2B_2}$	0.0417	0.0161
		$\Psi_{L_1B_2}$	0	0
		$\Psi_{L_2B_1}$	0	0
		$\sum$	0.0834	0.0846
Ψ_{BB}	0.0867	$\Psi_{B_1B_1}$	0.0385	0.0662
		$\Psi_{B_2B_2}$	0.0385	0.0143
		$\Psi_{B_1B_2}$	0.0101	0.0064
		$\sum$	0.0871	0.0869

Table 9: Numerical comparison of family type distributions.

	$\partial/\partial X_H$	$\partial/\partial X_{L_i}$	$\partial/\partial X_{B_i}$	$\partial/\partial X_{B_j}$	$\partial/\partial X_{L_j}$
$q_H(HB_i)$	0	≤ 0	0	0	0
$q_H(B_iB_i)$	≥ 0	≤ 0	0	≥ 0	0
$q_H(B_iB_j)$	0	≤ 0	0	0	≤ 0
$q_{L_i}(L_iB_i)$	≤ 0	0	0	≤ 0	0
$q_{L_i}(B_iB_i)$	≤ 0	≥ 0	0	≤ 0	0
$q_{B_i}(HB_i)$	0	≥ 0	0	0	0
$q_{B_i}(L_iB_i)$	≥ 0	0	0	≥ 0	0
$q_{B_i}(B_iB_i)$	≥ 0	≥ 0	0	≥ 0	0
$q_{B_i}(B_iB_j)$	0	≥ 0	0	0	≤ 0

Table 10: Signum of the partial derivatives of $q_L(F)$ for the nL model.

As for the other models, we assume that not all families are successful in their transmission efforts. Let $Q_R(F)$ denote the fraction of all F-type families who successfully transmit repertoire R to their children. We have

$$Q_H(HH) = Q_{L_i}(L_i, L_i) = 1,$$

and, analogously to the 1L models,

$$Q_H(HB_i; C, S, X) = (1 - C_{1,i}) \cdot q_H(HB_i; S, X) + C_{1,i}/2, \tag{5.182}$$
$$Q_H(B_iB_i; C, S, X) = (1 - C_{1,i}) \cdot q_H(B_iB_i; S, X), \tag{5.183}$$
$$Q_{L_i}(L_iB_i; C, S, X) = (1 - C_{1,i}) \cdot q_{L_i}(L_iB_i; S, X) + C_{1,i}, \tag{5.184}$$
$$Q_{L_i}(B_iB_i; C, S, X) = (1 - C_{1,i}) \cdot q_{L_i}(B_iB_i; S, X) + C_{1,i}. \tag{5.185}$$

Moreover, in the case of families with bilinguals speakers of different minority languages we set

$$Q_H(B_iB_j; C, S, X) = \frac{2 - C_{1,i} - C_{1,j}}{2} q_H(B_iB_j; S, X), \tag{5.186}$$
$$Q_{B_i}(B_iB_j; C, S, X) = \frac{2 - C_{1,i} - C_{1,j}}{2} q_{B_i}(B_iB_j; S, X) + \frac{C_{1,i}}{2}, \tag{5.187}$$
$$Q_{B_j}(B_iB_j; C, S, X) = \frac{2 - C_{1,i} - C_{1,j}}{2} q_{B_j}(B_iB_j; S, X) + \frac{C_{1,j}}{2}. \tag{5.188}$$

For $C_{1,i} = C_{1,j} = 0$, we have $Q_{LR}(B_iB_j) = q_{LR}(B_iB_j)$. For $C_{1,i} = C_{1,j} = 1$, both minorities are fully segregated with respect to the H-speaking majority population, and bilinguals do not raise their children as H-monolinguals. Since only concentration with respect to the majority language is taken into account (C_1 but not C_2), in this case half of the couples raise their children as B_i's and the other half raise their children as B_j's.

5.4.2.1 Functional form

Next, we offer a functional form of the $q_R(F)$ functions similar to the functional expressions presented for the 1L model. Let $h : [0,1]^2 \to [0,1]$ be a non-decreasing function in both dimensions with $h(0,0) = 0$ and $h(1,1) = 1$, and $\varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2$ be constants between 0 and 1 satisfying $\varepsilon_1 + \varepsilon_2 = 1$, $\zeta_2 \leq \zeta_1$ and $\zeta_1 + \zeta_2 \leq 1$. For the functional expressions we consider

$$q_H(HB_i; S, X) = \varepsilon_1\big(1 - h(X_{L_i}, S_i)\big) + \varepsilon_2 S_H, \tag{5.189}$$

$$q_H(B_iB_i; S, X) = \zeta_1\big(1 - h(X_{L_i}, S_i)\big) + \zeta_2 h\left(X_H + \sum_{j\neq i} X_{B_j}, S_H\right), \tag{5.190}$$

$$q_{L_i}(L_iB_i; S, X) = \varepsilon_1\left(1 - h\left(X_H + \sum_{j\neq i} X_{B_j}, S_H\right)\right) + \varepsilon_2 S_{L_i}, \tag{5.191}$$

$$q_{L_i}(B_iB_i; S, X) = \zeta_1\left(1 - h\left(X_H + \sum_{j\neq i} X_{B_j}, S_H\right)\right) + \zeta_2 h(X_{L_i}, S_i), \tag{5.192}$$

$$q_H(B_iB_j; S, X) = \frac{1}{2}\big(1 - h(X_{L_i}, S_i)\big) + \frac{1}{2}\big(1 - h(X_{L_j}, S_j)\big). \tag{5.193}$$

Note, $q_{B_i}(HB_i)$, $q_{B_i}(L_iB_i)$ and $q_{B_i}(B_iB_i)$ are fully determined by the above specifications, while $q_{B_i}(B_iB_j) + q_{B_j}(B_iB_j) = 1 - q_H(B_iB_j)$. We set

$$q_{B_i}(B_iB_j) = \frac{h(X_{L_i}, S_i)}{h(X_{L_i}, S_i) + h(X_{L_j}, S_j)}(1 - q_H(B_iB_j)) = \frac{h(X_{L_i}, S_i)}{2}.$$

These q-functions satisfy the conditions derived from the utility maximization approach, cf. Table 10.

5.4.3 Language dynamics

In the nL model, we have sets of n parameters for schooling and adult language learning. Consider

- s_{H,B_i}: fraction of children entering school as H monolinguals who leave school as B_i's,

- s_{L_i,B_i}: fraction of children entering school as L_i monolinguals who leave school as B_i's,
- $s_{B_i,H}$: fraction of children entering school as B_i's who leave school as H monolinguals,
- a_{H,B_i}: annual rate at which H monolinguals become B_i's,
- a_{L_i,B_i}: annual rate at which L_i monolinguals become B_i's.

As before, cf. (5.95)-(5.96), a possible functional form for a_{R_1,R_2} is given by

$$a_{H,B_i}(S, C; X) = (1 - C_{1,i}) \max\{0, \theta S_i X_{L_i} - \phi(1 - v_{L_i}), \tag{5.194}$$

$$a_{L_i,B_i}(S, C; X) = (1 - C_{1,i}) \max\left\{0, \theta S_H \left(X_H + \sum_{j \neq i} X_{B_j}\right) - \phi(1 - v_H)\right\}. \tag{5.195}$$

To describe the dynamics in the nL model, we again use functions f_R and g_R. For $R = H, L_1, ..., L_n$ we define

$$f_R(X) = \sum_F Q_R(F; C, S, X)\Psi(F; C, X)$$

and

$$g_H(X) = \left(1 - \sum_{i=1}^n s_{H,B_i}\right) f_H(X) + \sum_{i=1}^n s_{B_i,H} f_{B_i}(X), \tag{5.196}$$

$$g_{L_i}(X) = (1 - s_{L_i,B_i}) f_{L_i}(X), \tag{5.197}$$

$$g_{B_i}(X) = (1 - s_{B_i,H}) f_{B_i}(X) + s_{H,B_i} f_H(X) + s_{L_i,B_i} f_{L_i}(X). \tag{5.198}$$

The language dynamics is described by the differential equations

$$\dot{N}_H(t) = -\left[\mu + (1 - \mu)\sum_{i=1}^n a_{H,B_i}(t)\right] N_H(t) + \lambda N(t) g_H(X(t)) + M_H(t), \tag{5.199}$$

$$\dot{N}_{L_i}(t) = -\left[\mu + (1 - \mu) a_{L_i,B_i}\right] N_{L_i}(t) + \lambda N(t) g_{L_i}(X(t)) + M_{L_i}(t), \tag{5.200}$$

$$\begin{aligned}\dot{N}_{B_i}(t) = &-\mu N_{B_i}(t) + (1 - \mu) a_{H,B_i} N_H(t) + (1 - \mu) a_{L_i,B_i} N_{L_i}(t) \\ &+ \lambda N(t) g_{B_i}(X(t)) + M_{B_i}(t),\end{aligned} \tag{5.201}$$

$$\dot{N}(t) = (\lambda - \mu) N(t) + M(t). \tag{5.202}$$

5.4.3.1 Special case: constant migration flow

For $\lambda < \mu$, the population size N converges to $N(\infty) = M/(\mu - \lambda)$. In the steady state we obtain

$$\dot{X}_H(t) = -\left[\mu + (1-\mu)\sum_{j=1}^{n} a_{H,B_j}(t)\right] X_H(t) + \lambda g_H(X(t)) + (\mu - \lambda) M_H(t)/M(t), \tag{5.203}$$

$$\dot{X}_{L_i}(t) = -\left[\mu + (1-\mu) a_{L_i,B_i}(t)\right] X_{L_i}(t) + \lambda g_{L_i}(X(t)) + (\mu - \lambda) M_{L_i}(t)/M(t), \tag{5.204}$$

$$\dot{X}_{B_i}(t) = -\mu X_{B_i}(t) + (1-\mu) a_{H,B_i} X_H(t) + (1-\mu) a_{L_i,B_i} X_{L_i}(t) + \lambda g_{B_i}(X(t)) + (\mu - \lambda) M_{B_i}(t)/M(t). \tag{5.205}$$

If $\lambda \geq \mu$ and $M > 0$, then the population size diverges ($N(t) \to \infty$). In this case we get in the limit that

$$\dot{X}_H(t) = -\left[\lambda + (1-\mu)\sum_{j=1}^{n} a_{H,B_j}(t)\right] X_H(t) + \lambda g_H(X(t)), \tag{5.206}$$

$$\dot{X}_{L_i}(t) = -\left[\lambda + (1-\mu) a_{L_i,B_i}(t)\right] X_{L_i}(t) + \lambda g_{L_i}(X(t)), \tag{5.207}$$

$$\dot{X}_{B_i}(t) = -\lambda X_{B_i}(t) + (1-\mu) a_{H,B_i} X_H(t) + (1-\mu) a_{L_i,B_i} X_{L_i}(t) + \lambda g_{B_i}(X(t)). \tag{5.208}$$

5.5 Numerical comparison

In this section we compare the four models (1G-1L, 3G-1L, 10G-1L, 1G-nL) numerically. We consider numerical examples to illustrate similarities and differences between the models. We only consider the special case of a constant migration flow and use the functional expressions suggested in the previous sections. All models were implemented in MATLAB, and the numerical results are depicted in tables and figures.

5.5.1 Models with a single minority language

We want to compare projections for comparable scenarios produced by the four different models. To do so, we use the empirical examples studied in Templin (2018) and Templin (2019). In Templin (2018), we studied English (H) and Spanish (L) in the United States and in Templin (2019) Spanish (H) and Basque (L) in the Basque Autonomous Communities (BAC) in Spain. In both essays, we roughly estimated parameters from empirical data and applied a 1G-1L model to analyze both cases. The parameters and the numerical characterization of the linguistic environment for 1G-1L models are displayed in Table 11.

Not all the parameters displayed in Table 11 do actually appear in Templin (2018) and Templin (2019). In Templin (2018), we used a different functional form to model the language transmission fractions $q_R(F)$. In Templin (2019), these fractions were even considered to be constant. Here, we use the new functional form

Parameters / Environment		**Example 1** US	**Example 2** BAC
Rates	λ	0.015	0.0085
	μ	0.0087	0.0088
Status	S_H	0.75	0.6
	S_L	0.25	0.4
Transmission	$\varepsilon_1;\ \varepsilon_2$	0.6; 0.4	0.5; 0.5
	$\zeta_1;\ \zeta_2$	0.6; 0.3	0.6; 0.3
Concentration	C_1	0.27	0.47
	C_2	0	0
Schooling	$s_{L,B}$	1	1
	$s_{H,B}$	0.009	0.2
	$s_{B,H}$	0.59	0.24
Adult learning	$\theta;\ \phi$	$0.095;\ 0.2$	$0.095;\ 0.2$
	$v_H;\ v_L$	$0.8;\ 0$	$0.8;\ 0.5$
Migration	M_H	0	2,300
	M_L	626,000	0
	M_B	47,000	0
Initial composition	$N_H(0)$	187,187,000	1,605,700
	$N_L(0)$	5,372,000	0
	$N_B(0)$	5,724,000	509,800

Table 11: Model parameters for the two examples (1G-1L).

for $q_R(F)$ proposed in Section 5.2.3 and with new values for $\varepsilon_1, \varepsilon_1, \zeta_1, \zeta_2$. Moreover, adult language learning rates a_{R_1,R_2} were also assumed to be constant in Templin (2019). Here, we use the functional form for a_{R_1,R_2} that we outlined above, cf. Section 5.2.4.2 and that was also applied in Templin (2018). For the second example, we copy the values for θ, ϕ, v_H from first one, but set $v_L = 0.5$, since the learning of Basque (L) is supported by language policies in the BAC. Furthermore, there are no status variables in Templin (2019). We set $S_L = 0.4$ in the second example. Lastly, we assumed constant relative migration in Templin (2019) – i.e. $M(t)/N(t) \equiv const$ – instead of constant absolute migration – $M(t) \equiv const$ – considered here. To get constant migration numbers, we use the migration numbers at the middle of the time span analyzed in Templin (2019), i.e. $M(t) \equiv M(T/2)$, where $T/2$ corresponds to the year 2001. The aim of this section is not to analyze English/Spanish competition in the US or Spanish/Basque competition in the BAC. We just build on the empirical case studies in Templin (2018) and Templin (2019) to obtain two realistic numerical examples, which we can analyze to compare the different models.

Naturally, for the three 1L models (1G-1L, 3G-1L, 10G-1L), almost all model parameters are identical. One difference concerns population dynamics. In the 1G model, the annual birth rate λ and the annual death rate μ concern the whole population. So if N denotes the overall population size, then within one year λN children are born and μN people die. In the 3G and 10G model, we assume that only younger adults have children and that only older adults die. Accordingly, the rates $\lambda_{\mathcal{X}}$ and $\mu_{\mathfrak{X}}$ only concern younger adults, resp. older adults. Within one year, $\lambda_{\mathcal{X}}\mathcal{N}$ children are born and $\mu_{\mathfrak{X}}\mathfrak{N}$ people die, $\mathcal{N}$ and $\mathfrak{N}$ denoting the number of younger and older adults. Therefore, we have to have $\lambda_{\mathcal{X}} > \lambda$ and $\mu_{\mathfrak{X}} > \mu$, so that the number of births and deaths is similar and hence the models are comparable. Additional to that, we can not use the same death rates for the 3G-1L and the 10G-1L model, if both are to be compared. The reason for that is that due to the construction of both models, in the 3G-1L model every year $\mu_{\mathfrak{X}}\mathfrak{N}$ people die. In contrast, in the 10G-1L model every year $\mu_{\mathfrak{X}}\mathfrak{N}+(1-\mu_{\mathfrak{X}})N_{10}/10$ people die, since we do not consider individuals older than 100 years of age. Therefore, the death rate in the 10G-1L model has to be lower, such that in both models a comparable fraction of the population dies.

In the 1G model, we do not consider the age of migrants. To compare the different models, we have to assume that $M_R = m_R + \mathcal{M}_R + \mathfrak{M}_R$ and $m_R = \sum_{i=1}^{2} M_{i,R}$, $\mathcal{M}_R = \sum_{i=3}^{5} M_{i,R}$, $\mathfrak{M}_R = \sum_{i=6}^{10} M_{i,R}$, for $R = H, L, B$. For simplicity, we set $M_{1,R} = M_{2,R} = m_R/2$, $M_{3,R} = M_{4,R} = M_{5,R} = \mathcal{M}_R/3$ and $M_{6,R} = ... = M_{10,R} = \mathfrak{M}_R/5$. Last, we have to adapt the initial composition at time $t = 0$. As the aim of this chapter is to compare the different models and not to analyze the actual developments in the US and the BAC, we use rough estimates of the age distribution for both cases.[57] We assume that the age distribution is the same for all three language repertoire groups. For the US, we set $n_R = 0.25N_R$, $\mathcal{N}_R = 0.40N_R$ and $\mathfrak{N} = 0.35N_R$. For the BAC, we set $n_R = 0.2N_R$, $\mathcal{N}_R =$

[57] Estimates are based on estimated data for 2017, see https://www.cia.gov/library/publications/the-world-factbook/fields/2010.html

$0.43N_R$ and $\mathfrak{N} = 0.37N_R$. As for migration, we set $N_{1,R} = N_{2,R} = n_R/2$, $N_{3,R} = N_{4,R} = N_{5,R} = \mathcal{N}_R/3$ and $N_{6,R} = ... = N_{10,R} = \mathcal{N}_R/5$.

5.5.1.1 Numerical projections

We compare model projections for a time-span of $T = 50$ years. We consider the evolution of the linguistic composition as projected by the different models. Figures 15-16 display projections for the parameter constellations of Example 1 defined above. Figure 17 shows projections for Example 2. In Tables 13, 14 and 15 we compare the projections at different points in time. Note, in Example 1, the overall population size tends to ∞, while in Example 2, the overall population size tends towards a finite steady state.

Example 1

It can be seen in Figures 15 and 16 that the 1G-1L, the 3G-1L and the 10G-1L model produce comparable projections for the first 50 years. The overall population grows fastest in the 10G-1L model and slowest in the 1G-1L model. At the same time, the highest number of people speaking the minority language Spanish is projected in the 3G-1L model. In the 1G-1L model, the number of Spanish monolinguals is lower than in the 10G-1L model at $t = 50$, while the number of bilinguals is higher in the 1G-1L model compared to the 10G-1L model. But absolute numbers are to handle with caution, at least in the above examples. One reason for that are the birth and death rates. First of all, we assumed that only younger adults have children, although some children are also born to parents younger than 20 years of age and older than 50 years of age. Although we adapted the birth rates accordingly, this distorts the actual dynamics, since the rate $\lambda_{\mathcal{X}}$ only concerns the number of younger adults, but not the number of older children (15-19). If the age structure changes significantly, then the estimated and adapted rates do not yield proper results anymore. We have a similar problem with the assumption that only older adults die. Moreover, we made the simplifying assumption of an equal death rate for all older age groups in the 10G-1L model. It would be more realistic to work with specific death rates for each of the age groups, since death rates are higher for people in their nineties than for people in their fifties. For even more realistic 3G-1L and 10G-1L models, one should use specific birth and death rates for all age groups.

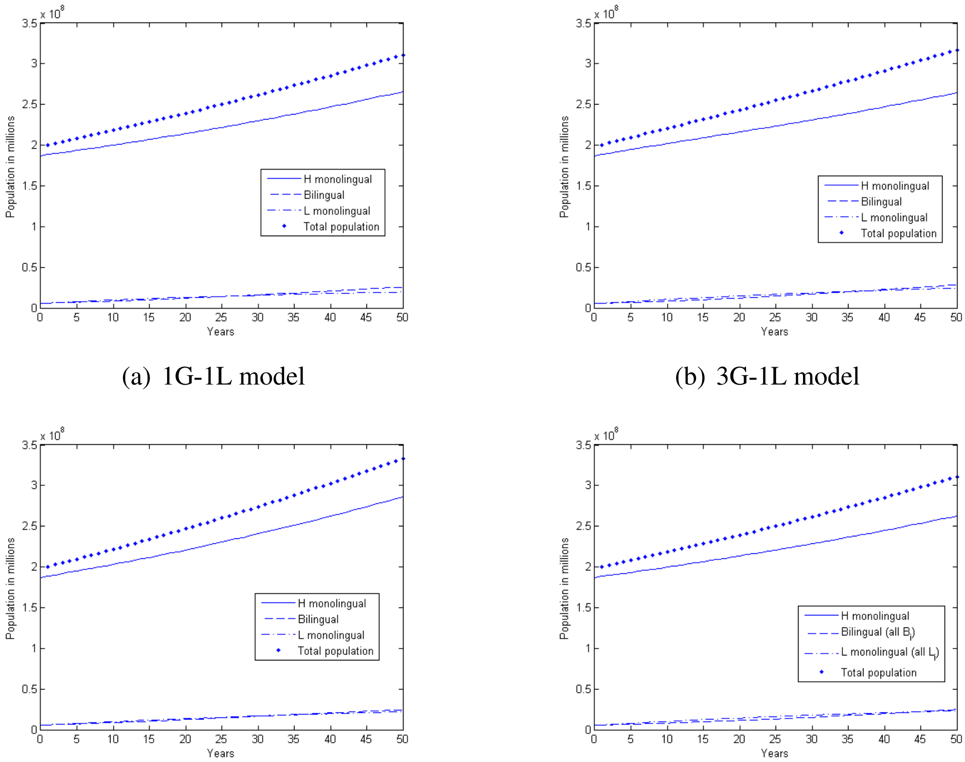
Considering relative instead of absolute numbers, we can see that the difference between the three projections is more moderate. After 50 years, 85.5% of the population is monolingual in H in the 1G-1L projection, 83.4% in the 3G-1L projection and 85.6% in the 10G-1L projection. So over a relatively long time span, differences are in the range of just 2%. This is not to say that 2% of a population of several hundred million people is not a relevant number. For a time span of 25 years the projections are even closer to one another, especially the 3G-1L and the 10G-1L projection. For both models, the projected fractions of bilinguals is equal and the fraction of Spanish monolinguals only differs by 0.3%, cf. Table 14. Since we assumed the same birth rate for all younger adults and the same death rate for all older adults, this similarity between the 3G-1L and

Parameters / Environment			**Example 1** US	**Example 2** BAC
Rates		$\lambda_{\mathcal{X}}$	0.0417	0.0198
		$\mu_{\mathfrak{X}}$	0.0249	0.0238
Migration	**H**	m_H	0	575
		$\mathcal{M}_H$	0	1,725
		$\mathfrak{M}_H$	0	0
	L	m_L	156,500	0
		$\mathcal{M}_L$	469,500	0
		$\mathfrak{M}_L$	0	0
	B	m_B	11,750	0
		$\mathcal{M}_B$	35,250	0
		$\mathfrak{M}_B$	0	0
Initial composition	**H**	$n_H(0)$	46,796,750	321,140
		$\mathcal{N}_H(0)$	74,874,800	690,451
		$\mathfrak{N}_H(0)$	65,515,450	594,109
	L	$n_L(0)$	1,343,000	0
		$\mathcal{N}_L(0)$	2,148,800	0
		$\mathfrak{N}_L(0)$	1,880,200	0
	B	$n_B(0)$	1,431,00	101,960
		$\mathcal{N}_B(0)$	2,289,600	219,214
		$\mathfrak{N}_B(0)$	2,003,400	188,626

Table 12: Model parameters for the two examples (3G-1L).
Note, $0.015/(0.4 \cdot 0.9) = 0.0417$, $0.0085/0.43 = 0.0198$, $0.0087/0.35 = 0.0249$ and $0.0088/0.37 = 0.0238$. Migration numbers were obtained via $m_R = 1/4 * M_R$ and $\mathcal{M}_R = 3/4 * M_R$, i.e. for the examples it is assumed that only children and younger adults migrate. The rates $\lambda_{\mathcal{X}}$ are additionally adapted: In 2016, though, about 10% of all children in the US were born to mothers younger than 20 years of age. In Spain this number is negligible. Cf.
https://www.statista.com/statistics/241533/birth-rate-by-age-of-mother-in
-the-us/
and
https://www.statista.com/statistics/449605/number-of-births-in-spain-by
-age-of-mother/

		$t=0$	$t=10$	$t=25$	$t=50$
1G-1L	$\mathbf{N_H(t)}$	187,187	200,087	221,882	265,763
	$\mathbf{N_L(t)}$	5,372	9,591	14,292	19,500
	$\mathbf{N_B(t)}$	5,724	8,380	13,977	25,574
3G-1L	$n_H(t)$	46,796	53,476	60,585	72,338
	$\mathcal{N}_H(t)$	74,874	75,235	80,082	93,607
	$\mathfrak{N}_H(t)$	65,515	73,245	82,832	98,532
	$\mathbf{N_H(t)}$	187,187	201,957	223,500	264,479
	$n_L(t)$	1,343	2,818	4,867	7,497
	$\mathcal{N}_L(t)$	2,148	5,017	7,440	9,274
	$\mathfrak{N}_L(t)$	1,880	2,541	4,423	7,719
	$\mathbf{N_L(t)}$	5,372	10,377	16,731	24,491
	$n_B(t)$	1,431	1,967	3,389	6,140
	$\mathcal{N}_B(t)$	2,289	3,868	7,222	13,383
	$\mathfrak{N}_B(t)$	2,003	2,442	3,984	8,706
	$\mathbf{N_B(t)}$	5,724	8,278	14,596	28,230
10G-1L	$n_H(t)$	46,796	54,496	62,761	77,741
	$\mathcal{N}_H(t)$	74,874	74,475	80,385	100,342
	$\mathfrak{N}_H(t)$	90,473	74,261	87,232	108,207
	$\mathbf{N_H(t)}$	212,145	203,234	230,379	286,292
	$n_L(t)$	1,343	2,099	2,923	3,529
	$\mathcal{N}_L(t)$	2,148	4,893	6,708	7,420
	$\mathfrak{N}_L(t)$	2,596	2,699	5,605	10,990
	$\mathbf{N_L(t)}$	6,088	9,693	15,237	21,940
	$n_B(t)$	1,431	2,483	4,446	6,763
	$\mathcal{N}_B(t)$	2,289	3,694	6,022	9,057
	$\mathfrak{N}_B(t)$	2,766	2,444	4,025	8,599
	$\mathbf{N_B(t)}$	6,487	8,622	14,494	24,420

Table 13: Projections for Example 1 – 1G-1L, 3G-1L and 10G-1L. Numbers are given in thousends.



(a) 1G-1L model

(b) 3G-1L model

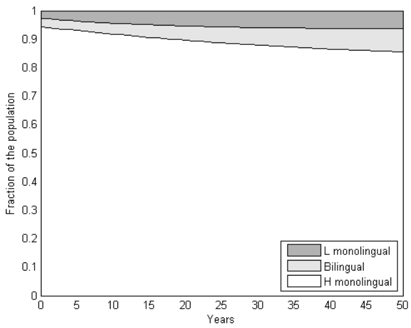
(c) 10G-1L model

(d) 1G-2L model

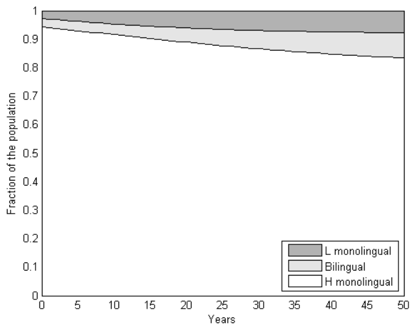
Figure 15: Projections for Example 1 – Evolution of the numbers of monolingual and bilingual speakers.

		$t=0$	$t=10$	$t=25$	$t=50$
1G-1L	$\mathbf{X_H(t)}$	0.944	0.918	0.887	0.855
	$\mathbf{X_L(t)}$	0.027	0.044	0.057	0.063
	$\mathbf{X_B(t)}$	0.029	0.038	0.056	0.082
3G-1L	$x_H(t)$	0.944	0.918	0.880	0.841
	$\mathcal{X}_H(t)$	0.944	0.894	0.845	0.805
	$\mathfrak{X}_H(t)$	0.944	0.936	0.908	0.857
	$\mathbf{X_H(t)}$	0.944	0.915	0.877	0.834
	$x_L(t)$	0.027	0.048	0.071	0.087
	$\mathcal{X}_L(t)$	0.027	0.060	0.079	0.080
	$\mathfrak{X}_L(t)$	0.027	0.032	0.048	0.067
	$\mathbf{X_L(t)}$	0.027	0.047	0.066	0.077
	$x_B(t)$	0.029	0.034	0.049	0.071
	$\mathcal{X}_B(t)$	0.029	0.046	0.076	0.115
	$\mathfrak{X}_B(t)$	0.029	0.031	0.044	0.076
	$\mathbf{X_B(t)}$	0.029	0.038	0.057	0.089
10G-1L	$x_H(t)$	0.944	0.922	0.895	0.883
	$\mathcal{X}_H(t)$	0.944	0.897	0.863	0.859
	$\mathfrak{X}_H(t)$	0.944	0.923	0.885	0.840
	$\mathbf{X_H(t)}$	0.944	0.914	0.880	0.856
	$x_L(t)$	0.027	0.036	0.042	0.040
	$\mathcal{X}_L(t)$	0.027	0.059	0.072	0.064
	$\mathfrak{X}_L(t)$	0.027	0.043	0.068	0.088
	$\mathbf{X_L(t)}$	0.027	0.047	0.063	0.069
	$x_B(t)$	0.029	0.042	0.063	0.077
	$\mathcal{X}_B(t)$	0.029	0.044	0.065	0.078
	$\mathfrak{X}_B(t)$	0.029	0.033	0.047	0.072
	$\mathbf{X_B(t)}$	0.029	0.039	0.057	0.075

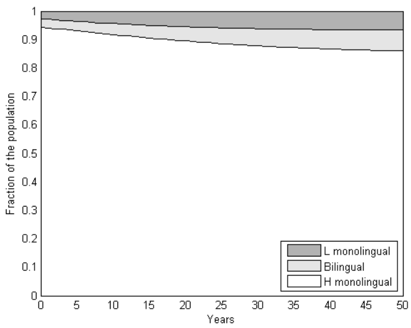
Table 14: Linguistic composition for Example 1 – 1G-1L, 3G-1L and 10G-1L.



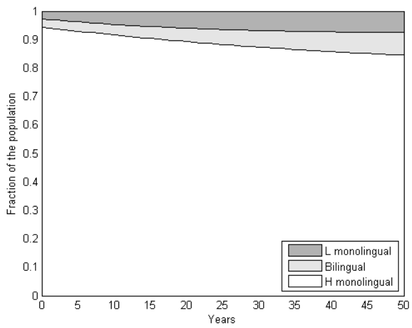
(a) 1G-1L model



(b) 3G-1L model



(c) 10G-1L model



(d) 1G-2L model

Figure 16: Projections for Example 1 – Evolution of the linguistic composition.

the 10G-1L model is what one would expect. In the 1G-1L model, the fraction of Spanish monolinguals after 25 years is almost 1% lower than in the 3G-1L model. One reason for that is the assumption that Spanish monolingual migrants are assumed to be children or younger adults in the 3G model. That implies that they are part of the population for a while until they finally grow old and die. In the 1G-1L model their age is not taken into account and so they could potentially die shortly after migrating.

Example 2
In Example 2, the most obvious difference between the projections of the 1G-1L model on the one hand and the 3G-1L and the 10G-1L model on the other hand is the existence of Basque monolingual speakers in the latter two. It can be seen in Table 15 that all Basque monolinguals are children. Some Basque speaking parents only speak Basque at home and so their children can be counted as Basque monolinguals, at least for the first years in their life. At the latest when reaching adulthood, they have acquired Spanish and are hence not monolingual anymore. Since the 1G-1L model only considers adults, these Basque monolingual children do not appear in the projections. If these children are considered as future bilinguals, the 1G-1L and the 3G-1L model yield almost identical projections. In contrast,the 10G-1L model projects higher fractions of bilinguals and hence lower fractions of Spanish monolinguals. The difference to the 1G-1L and 3G-1L model increases over time, but is still within a 2% range after 25 years.

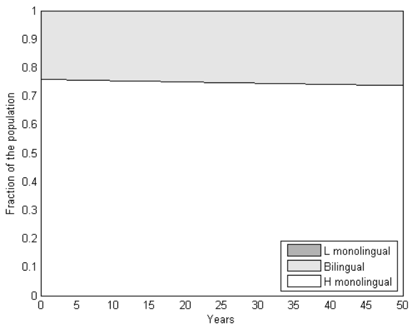
Summary
To summarize, it can be said that for moderate time spans the simple 1G-1L model yields a useful approximation for the more complex but also more realistic 3G-1L and 10G-1L model. If absolute numbers are relevant – e.g. for the estimation of costs and benefits of language policy measures –, then the age structure should be taken into account to obtain more realistic projections. Moreover, if sufficient data are available, one should use age group or cohort specific birth and death rate to obtain even more realistic projections with the 10G-1L model.

5.5.2 1G-1L vs. 1G-2L

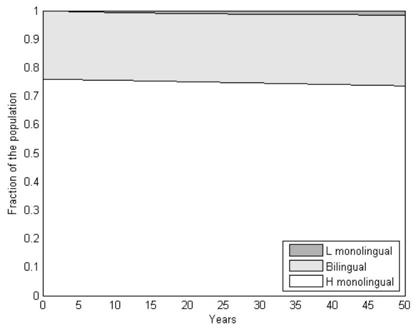
In this section, we investigate the effects of a second minority language on the dynamics. We consider the numerical Example 1 from the previous section, and consider two cases. In case A, both minority languages (L_1 and L_2) have identical features, i.e. same status, same size, same concentration, etc. In case B, we have a bigger minority language with a higher status and a smaller minority language with a lower status. Most of the associated parameters are displayed in Table 16. Parameters for language transmission ($\varepsilon_1, \varepsilon_1, \zeta_1, \zeta_2$) and adult language learning (θ, ϕ, v_H) as well as birth and migration rates are identical to those in the 1G-1L model. We assume that $v_{L_1} = v_{L_2}$. All other parameters and environment characteristics are displayed in Table 16. The status values were obtained as follows. Assume that the absolute status values in the 1L case are $\overline{S}_H = 6$ and $\overline{S}_L = 2$. Then the relative status values are $S_H = 0.75$ and $S_L = 0.25$, as in Example 1

		$t=0$	$t=10$	$t=25$	$t=50$
1G-1L	$\mathbf{X_H(t)}$	0.759	0.754	0.747	0.737
	$\mathbf{X_L(t)}$	0	0	0	0
	$\mathbf{X_B(t)}$	0.241	0.246	0.253	0.263
3G-1L	$x_H(t)$	0.759	0.778	0.789	0.790
	$\mathcal{X}_H(t)$	0.759	0.744	0.728	0.712
	$\mathfrak{X}_H(t)$	0.759	0.757	0.751	0.741
	$\mathbf{X_H(t)}$	0.759	0.756	0.750	0.740
	$x_L(t)$	0	0.034	0.067	0.096
	$\mathcal{X}_L(t)$	0	0	0	0
	$\mathfrak{X}_L(t)$	0	0	0	0
	$\mathbf{X_L(t)}$	0	0.006	0.010	0.011
	$x_B(t)$	0.241	0.188	0.144	0.114
	$\mathcal{X}_B(t)$	0.241	0.256	0.272	0.288
	$\mathfrak{X}_B(t)$	0.241	0.243	0.249	0.259
	$\mathbf{X_B(t)}$	0.241	0.238	0.241	0.250
10G-1L	$x_H(t)$	0.759	0.742	0.722	0.691
	$\mathcal{X}_H(t)$	0.759	0.736	0.694	0.649
	$\mathfrak{X}_H(t)$	0.759	0.759	0.749	0.717
	$\mathbf{X_H(t)}$	0.759	0.748	0.729	0.696
	$x_L(t)$	0	0.029	0.049	0.066
	$\mathcal{X}_L(t)$	0	0	0	0
	$\mathfrak{X}_L(t)$	0	0	0	0
	$\mathbf{X_L(t)}$	0	0.001	0.007	0.008
	$x_B(t)$	0.241	0.229	0.229	0.244
	$\mathcal{X}_B(t)$	0.241	0.264	0.306	0.351
	$\mathfrak{X}_B(t)$	0.241	0.241	0.251	0.283
	$\mathbf{X_B(t)}$	0.241	0.247	0.264	0.296

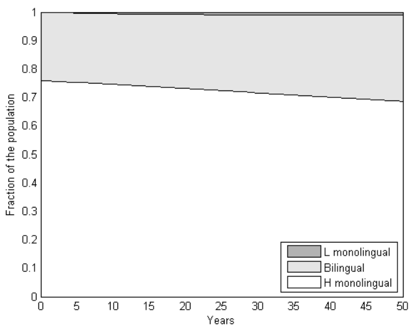
Table 15: Linguistic composition for Example 2 – 1G-1L, 3G-1L and 10G-1L.



(a) 1G-1L model



(b) 3G-1L model



(c) 10G-1L model

Figure 17: Projections for Example 2 – Evolution of the linguistic composition.

above. Now we introduce a second minority language, with $\overline{S}_2 = 2$ in case A and $\overline{S}_2 = 1$ in case B. Then we obtain the relative status values shown in Table 16.

5.5.2.1 Numerical Projections

As before, we compare model projections for a time-span of $T = 50$ years. We look at the evolution of the linguistic composition as projected by the 1G-1L and the two 1G-2L models. In Tables 17 and 18 we compare the projections at different points in time. The evolution of the absolute and relative numbers is depicted graphically in Figure 18.

Only differentiating between those who speak a minority language and those who do not, Figure 18 shows that the 1G-1L and both 1G-2L models behave similarly. At every point in time, the overall population size is the same in all three models. Moreover, the sizes of all language groups increase for the first fifty years in all three models, cf. Table 17. What differs slightly are the relative sizes of the different language groups, see Table 18. After 25 years, the fractions X_H, X_L and X_B only differ by at most 0.5%, and the 1G-1L model projects the highest fraction of monolingual speakers of the majority language English. After 50 years, this number is 1% higher in the 1G-1L model compared to the two 1G-2L models. One reason for that is that in the 1G-1L example the relative status of English is higher than in the two cases with two minority languages. For the same reason, the fraction of monolingual speakers of a minority language $X_L = X_{L_1} + X_{L_2}$ is higher in 1G-2L models.

It can also be seen that in the symmetrical case A, both minority languages evolve identically. This is exactly what one would expect, since both minority languages have the exact same features. In the asymmetrical case B, the weaker minority language L_2 gains speakers as the stronger minority language L_1 – at least for the first fifty years –, but at a lower pace. Moreover, there are more bilinguals than monolinguals after 50 years in both languages in both cases, but in case B the difference is higher. This is due to the fact that in case B fewer L_2 monolinguals migrate to the population and that fewer families solely transmit the low status language L_2 to their children.

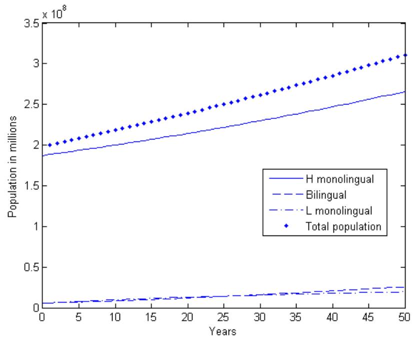
As for the models with several age groups, we can say that the 1G-1L models yields a useful approximation for the more complicated case of several minority languages. Nonetheless, if one wants to analyze language policies directed at single linguistic minorities, a model that accounts for the full linguistic diversity of the population is called for.

5.6 Conclusions

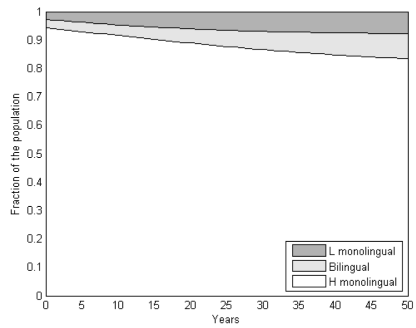
In this essay, we presented and analyzed extensions of a basic language dynamics model developed in previous research. For a majority language H and a single minority language L, the original model describes how the numbers of monolingual speakers of H and L as well as the number of bilinguals change over time. It models five central processes of language change, namely population dynamics (birth,

Parameters / Environment		Example 1 **Case A**	**Case B**
Status	S_H	0.6	2/3
	S_{L_1}	0.2	2/9
	S_{L_2}	0.2	1/9
Concentration	$C_{1,1}$	0.27	0.27
	$C_{2,1}$	0	0
	$C_{1,2}$	0.27	0.27
	$C_{2,2}$	0	0
Schooling	s_{L_1,B_1}	1	1
	s_{H,B_1}	0.0045	0.009
	$s_{B_1,H}$	0.59	0.39
	s_{L_2,B_2}	1	1
	s_{H,B_2}	0.0045	0
	$s_{B_2,H}$	0.59	0.79
Migration	M_H	0	0
	M_{L_1}	313,000	469,500
	M_{B_1}	23,500	35,250
	M_{L_2}	313,000	156,500
	M_{B_2}	23,500	11,750
Initial composition	$N_H(0)$	187,187,000	187,187,000
	$N_{L_1}(0)$	2,686,000	4,029,000
	$N_{B_1}(0)$	2,862,000	4,293,000
	$N_{L_2}(0)$	2,686,000	1,343,000
	$N_{B_2}(0)$	2,862,000	1,431,000

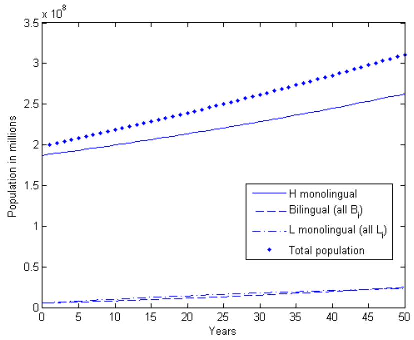
Table 16: Model parameters for the two examples (1G-2L).



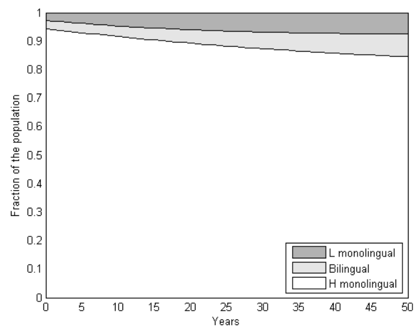
(a) 1G-1L model



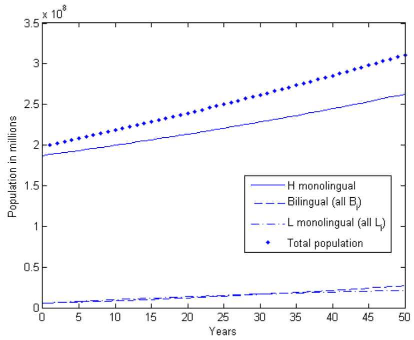
(b) 1G-1L model



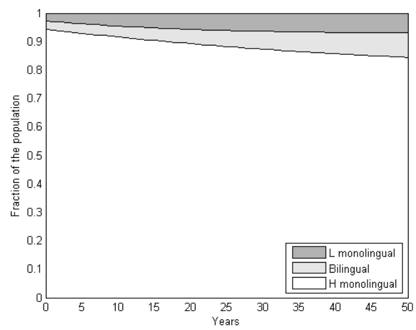
(c) 1G-2L model, Case A



(d) 1G-2L model, Case A



(e) 1G-2L model, Case B



(f) 1G-2L model, Case B

Figure 18: Projections for Example 1, Case A and Case B.

		$t = 0$	$t = 10$	$t = 25$	$t = 50$
1G-1L	$\mathbf{N_H(t)}$	187,187	200,087	221,882	265,763
	$\mathbf{N_L(t)}$	5,372	9,591	14,292	19,500
	$\mathbf{N_B(t)}$	5,724	8,380	13,977	25,574
1G-2L - Case A	$\mathbf{N_H(t)}$	187,187	199,790	220,809	262,721
	$N_{L_1}(t)$	2,686	5,101	8,056	11,740
	$N_{L_2}(t)$	2,686	5,101	8,056	11,740
	$\mathbf{N_L(t)}$	5,372	10,202	16,112	23,480
	$N_{B_1}(t)$	2,862	4,033	6,615	12,318
	$N_{B_2}(t)$	2,862	4,033	6,615	12,318
	$\mathbf{N_B(t)}$	5,724	8,066	13,230	24,637
1G-2L - Case B	$\mathbf{N_H(t)}$	187,187	199,754	220,736	262,635
	$N_{L_1}(t)$	4,029	7,445	11,461	16,244
	$N_{L_2}(t)$	1,344	2,468	3,762	5,212
	$\mathbf{N_L(t)}$	5,372	9,913	15,224	21,457
	$N_{B_1}(t)$	4,293	6,397	10,931	20,770
	$N_{B_2}(t)$	1,431	1,994	3,258	5,976
	$\mathbf{N_B(t)}$	5,724	8,391	14,190	26,746

Table 17: Projections for Example 1, Case A and Case B. Numbers in thousends.

		$t=0$	$t=10$	$t=25$	$t=50$
1G-1L	$\mathbf{X_H(t)}$	0.944	0.918	0.887	0.855
	$\mathbf{X_L(t)}$	0.027	0.044	0.057	0.063
	$\mathbf{X_B(t)}$	0.029	0.038	0.056	0.082
1G-2L Case A	$\mathbf{X_H(t)}$	0.944	0.916	0.883	0.845
	$X_{L_1}(t)$	0.014	0.023	0.032	0.038
	$X_{L_2}(t)$	0.014	0.023	0.032	0.038
	$\mathbf{X_L(t)}$	0.027	0.047	0.064	0.076
	$X_{B_1}(t)$	0.014	0.018	0.026	0.040
	$X_{B_2}(t)$	0.014	0.018	0.026	0.040
	$\mathbf{X_B(t)}$	0.029	0.037	0.053	0.079
1G-2L Case B	$\mathbf{X_H(t)}$	0.944	0.916	0.882	0.845
	$X_{L_1}(t)$	0.020	0.034	0.046	0.052
	$X_{L_2}(t)$	0.007	0.011	0.015	0.017
	$\mathbf{X_L(t)}$	0.027	0.045	0.061	0.069
	$X_{B_1}(t)$	0.022	0.029	0.044	0.067
	$X_{B_2}(t)$	0.007	0.009	0.013	0.019
	$\mathbf{X_B(t)}$	0.029	0.038	0.057	0.086

Table 18: Projections for Example 1, Case A and Case B.

death and migration), family formation, language transmission within the family, language education and adult language learning. To do so, several aspects of the linguistic environment are taken into account, for example the linguistic composition of the population, the status of the languages in question, spatial concentration of speakers of the minority language and language education policies.

The essay starts with a presentation of a slightly generalized version of the basic model. The generalized basic model treats all individuals with the same language repertoire equally. For the first two extensions of the basic model, we differentiate individuals with the same language repertoire along an age dimension. Since no such differentiation can be found in the basic model, it is called 1G-1L (one generation - one minority language). For a first age sensitive extension, we considered a model with three generations (3G-1L): children (0-19 years), young adults (20-49 years) and old adults ($\geq$50 years). For a second extension, we considered ten 10-year age groups or cohorts. Both models were presented and steady states were analyzed. For a second type of extension of the basic 1G-1L model, we considered the case of a majority language H and multiple minority languages $L_1, ..., L_n$. It is shown how the formulas have to be adapted to account for more than one minority language, getting the 1G-1L model as a special case.

For a numerical comparison, all three extension as well as the basic model itself were implemented in MATLAB. We compared the models for two empirical cases studied in previous research: English and Spanish in the United States were studied in Templin (2018), and Spanish and Basque in the Basque Autonomous Communities studied in Templin (2019). We produced projections for both cases and all four models for a time span of 50 years. The numerical comparison illustrated two results. First, it showed that the easier 1G-1L model yields a useful approximation of the more complicated 3G-1L, 10G-1L and 1G-2L models. For moderate time spans, e.g. 25 years, the differences in the projections produced by the 1G-1L on the one hand and the 3G-1L and 10G-1L model on the other hand are – depending on what the model shall be used for – in an acceptable range. The difference increases over time, but stays within a 2% range after 25 years. And 25 years is a time span for which the assumption of constant model parameters, e.g. birth rates and migration numbers, could be acceptable, depending on the empirical case the model is used for. It could also be seen that the 1G-1L model captures the aggregated dynamics of a situation with two minority languages quite well, at least for moderate time spans. On the other hand, it could be seen with the 3G-1L and 10G-1L model projections, that the linguistic compositions of the different generations or cohorts evolve differently. If the models shall be used to estimate costs and benefits of language policies, cf. Templin (2019), targeting only certain age groups – e.g. language education policies –, then an age sensitive model yields a clear advantage over the simple 1G-1L model. Similarly, the projections produced by the 1G-2L model illustrate that different minority languages evolve differently. So if language policies only target one of the minority language groups, then for the estimation of costs and benefits of such policies the 1G-2L is to be preferred over the simple 1G-1L model.

In this essay, we made several simplifying assumption to streamline the mathematical models and their presentation. To obtain even more realistic projections, some model improvements are worthy to be taken into account. First, one could use age and language group specific birth and death rates, instead of assuming that the rates are the same for all groups. Furthermore, these rates do not have to be constant over time. If there are estimations on how the rates will evolve in the future, then such estimates, i.e. dynamic rates, could be incorporated in the models. A similar argument can be made for migration numbers. Moreover, the rate of adult language learning might depend on the age of adults or a spatial dimension that goes beyond concentration. For the 1G-nL model, we assumed that individuals can only speak one minority language. This could be extended to several minority languages so that the model can account for multilingual people. All such details could be incorporated in the models presented above. One major obstacle for for complex versions of the models presented above is their application to real life case studies, since they require more detailed empirical information. This is already the case for the presented extensions in comparison to the simple basic model. For many countries and regions in this world, empirical language related data are scarce. Even setting up the basic model is currently not possible in many cases. So there is a trade-off between model complexity and applicability.

6 General summary and outlook

The aim of the research presented in this thesis was to improve the analysis of language policies. Costs and/or benefits of language policies most often depend on the linguistic composition of the population subject to the policy. This linguistic composition is in constant flux. Consequently, costs and benefits of language policies are not constant but change over time with changes in the linguistic composition. These changes can be significant, but standard policy analysis methods do not account for them. Therefore, my research combines established policy analysis tools like cost-benefit analysis with models for language dynamics. Language dynamics models can produce projections of the future development of the linguistic composition. These projections can then be used to estimate future costs and benefits of the policy in question. Discounting the estimated future costs and benefits yields a more realistic estimation of the net present value of the policy than a net present value that is based on constant future costs and benefits.

The main ingredient of the dynamic language policy analysis approach is the mathematical model that produces future projections of the linguistic composition. Therefore, I dedicated a large part of this thesis to the design and analysis of adequate models. What is considered adequate depends on the type of analysis and the kind of policy statements one is interested in. There is, for example, a difference between an analysis at a general and abstract level, and an analysis of a particular policy in a real life scenario. Accordingly, the corresponding model is more abstract in the former case and has to be more specific in the latter. Independent of the application, models that ought to be used for dynamic language policy analysis have to satisfy two basic conditions: They should be able to reproduce observed historical data, and they should allow to model language policies in a meaningful way.

In all four essays presented in this thesis, I used different versions and extensions of the language competition model proposed in Wickström (2005). I called this model W. It mainly focuses on language transmission from one generation to the next. Parents are conceptualized as utility maximizing actors. On the one hand, they want to raise their children in a language with a wide communicative range. On the other hand, they gain utility from transmitting their heritage language. Hence, if they are speakers of a minority language in an environment dominated by a majority language, they have to weigh the communication aspect against the identity aspect. I extended model W in two ways.

First, I added several sociolinguistic processes relevant for language dynamics to the original model. The resulting model E^1 presented in Chapter 3 includes intergenerational language transmission, language education, adult language learning and migration. These are all crucial areas for policy intervention with respect to language. With E^1 I developed a model that can reconstruct and project empirical language dynamics more accurately and that enables an analysis of the effects of language policies on the linguistic composition.

The second extension concerns the number of minority languages as well as the age structure of the population. In model W as well as in the first extension E^1, only one minority language is considered, and no differentiation is made with respect to the age of individuals. In Chapter 5, I therefore presented a model with multiple minority languages (E^2_{nL}) as well as a model with multiple age groups (E^2_{mG}). These two models are to be preferred over the previous models if one wants to analyze language policies that target only one of several minority language groups or policies that only aim at certain age groups. The model with multiple age groups (E^2_{mG}) can also yield an advantage over a single age group model (e.g. E^1) if language skills are distributed unequally among age groups. This is, for example, the case if a declining minority language is mainly spoken by older people within a population. I also showed that model E^1 yields a useful approximation for the more complex models E^2_{nL} and E^2_{mG}, if all minority language groups are comparable in size and status, and if the distribution of language skills is distributed homogeneously with respect to age.

Wickström (2005) showed that stable bilingual steady states are possible in his basic model W, as long as the status of the minority language is sufficiently high. In Chapter 2, my coauthors and I considered the case of a declining minority language with a decreasing status (model $\widetilde{W}$). The state can counteract the decrease through investments into status planning. We proved that it is not only possible to reverse the decline of the minority language, but that this can even be optimal, as long as the state is interested in preserving the minority language at reasonable costs. We showed that if the objective is to maximize the number of bilinguals at lowest possible costs, then the optimal investment strategy can lead to a stable bilingual equilibrium. We also analyzed steady states for the extended model E^1 that incorporates inter-generational language transmission, language education, adult language learning, and migration. It could be shown that monolingual and bilingual stable steady states are possible. The status of the minority language as well as the relation between birth, death and migration rates prove to be crucial in this matter. The same kind of statements on steady states can be made for the second more complex extensions E^2_{mG} and E^2_{nL}.

The second ingredient of the dynamic language policy analysis approach is cost-benefit (or cost-effectiveness) analysis. To perform the dynamic analysis, it has to be specified how costs and benefits depend on the linguistic composition of the population. In Chapter 2, we assumed benefits to be proportional to the number of bilinguals. Costs increase with the amount of resources invested in status planning. They can also increase with the number of speakers of the minority language in a concave fashion. This way, different kinds of policies can be investigated. Since at every point in time the state can alter the size of the investment, the dynamic approach yields a non-trivial deterministic optimal control problem. We showed that if the minority language is valued high enough by the policy maker, then the optimally controlled language dynamics can tend to a bilingual steady state. We also solved the control problem numerically for a specific case to illustrate this behavior.

In Chapter 4, I explored the possibility of applying dynamic language policy analysis to specific policies in a real life context. If one wants to analyze real policy options, I argue, then the underlying language dynamics model has to be sufficiently complex and realistic. This was my main motivation to develop extensions E^1, E^2_{mG} and E^2_{nL}. To perform cost-benefit analysis, it has to be specified how both costs and benefits depend on the linguistic composition. What distinguishes the analysis in Chapter 4 from the more abstract one in the previous chapter is the way language policies are modeled. In Chapter 2, the state could decide on the amount of resources it invests into status planning at every point in time. The investment is a control or policy variable in the model. The model in Chapter 4 does not contain any explicit policy variable. There, I modeled language policies as a change in corresponding model parameters. Policies can range from increasing the number of languages on bank notes or street signs, introducing different language education programs, supporting language courses for migrant newcomers or putting up a new television program in a minority language. Language education policies, for example, are likely to alter the linguistic outcome of public education. Therefore, they can be modeled as a change in those parameters that correspond to language education in public schools. Hence, to apply dynamic language policy analysis, estimates on the effects of a policy under scrutiny on the model parameters have to be provided. I designed the model in such a way that its parameters reflect measurable statistical figures, so that experts and policy makers from the respective field can produce such estimates. I illustrated the application of dynamic language policy analysis to the case of a policy concerning the introduction of a Basque TV program in the Basque Autonomous Communities in northern Spain. I showed that a non-dynamic analysis can underestimate the future benefits of the policy, while the dynamic analysis yields a more accurate estimate. The difference between the non-dynamic and the dynamic analysis can be significant to the point where the non-dynamic analysis yields a negative net present value (do not implement the policy), while the more accurate dynamic analysis yields a positive net present value (implement the policy).

The analysis of the Basque TV program demonstrates the potential practical relevance of dynamic language policy analysis. I implemented the language competition model and estimated all model parameters from empirical macro data on Basque and Spanish in the Basque Autonomous Communities available online. Much of the data stem from Census like surveys or special surveys on minority language issues. Similarly, the majority of model parameters for the analysis of Spanish and English in the United States could be estimated from US Census data. I showed that in both empirical cases, the extended models (E^1 and E^1_{const}) equipped with parameters estimated from empirical data can reproduce past dynamics of the linguistic composition of the Basque Autonomous Communities and the US. I also demonstrated that projections for future dynamics can be utilized to evaluate actual empirical policy options, as long as policy makers and experts can specify the structure of costs and benefits and provide estimates on the effects of the policy on relevant model parameters. Hence, if implemented in an accessible way, dynamic language policy analysis can function as a practical tool for policy makers to inform their language policy decisions.

A major obstacle for the application of dynamic language policy analysis to empirical policies is the availability of data. To run the extended models one needs data on births and deaths, migration, language education and its performance. For the extended model with only empirically estimated parameters (model E^1_{const}), one additionally needs data on language transmission in families and adult language learning. I analyzed two cases for which such data could be found. For many other countries, comparable information is not available. If Census surveys in a country do not ask for languages at all or just for the mother tongue or first language, it is difficult to even get a realistic picture of the current linguistic composition of the country, let alone its future development. Such a lack of empirical information makes it almost impossible to perform sensible analysis of policies that possibly target millions of people. I therefore hope that the dynamic approach can provide additional arguments for states and statistical agencies to collect more data on language skills in multiple languages and other relevant sociolinguistic information.

This thesis can be seen as a point of departure for further application of and research on dynamic language policy analysis. On the one hand, the dynamic models can be applied to more case studies for which data are available. Researchers and practitioners can investigate the future evolution of the linguistic composition they are interested in, and compare the (welfare) effects of competing policy options. They are not bound to evaluate policies on the basis of cost-benefit analysis, but can also compare the cost-effectiveness in a similar dynamic fashion. On the other hand, future research can improve the mathematical models further to improve the accuracy and reliability of the projections they provide. Throughout the thesis, I assumed crucial model parameters to be constant. If these parameters themselves change over time, the constancy assumption limits the time span for which meaningful projections can be derived. The birth and death rates as well as migration numbers are especially relevant in this regard. Future research can develop models that incorporate increasing or decreasing rates. Such models would require additional empirical information. It is out of the scope of a single researcher to gather all necessary information on socio-linguistic processes, general population dynamics and migration to set up the model. It is the theoretical and empirical research from scholars from sociolinguistics, sociology, migration studies, demography and econometrics which informed the theoretical design of the models I developed and that makes their application possible. Language policy analysis is an interdisciplinarity endeavor. My research showed that the contribution of applied mathematics and welfare economics to this endeavor are rigorous frameworks for combining quantitative information from multiple sources to inform policy decisions.

Appendices

A Appendix Section 2

Now, all N pairs are chosen randomly after one another. The total expected number of R_1R_2-type pairs equals $N \cdot \mathbb{P}[R_1R_2]$ and hence the expected fraction of R_1R_2-type pairs is $\mathbb{P}[R_1R_2]$. After this first step we have N pairs with $2NX_HX_L$ of them being of type HL. Recall, we assume that parents shall be able to properly communicate with each other, and therefore we exclude HL families. Splitting these HL pairs again and repeating the random selection we obtain new pairs of types HH, LL and HL. This procedure is repeated until only HH and LL pairs are left. This way, half of the $2NX_HX_L$ pairs of type HL will be transformed into HH pairs, while the other half will form LL pairs. As a result, we obtain the numbers presented in Table 1. Note, due to the law of large numbers (N is assumed to be large), the realized number of R_1R_2-type of pairs can be approximated by the expected number.

A.1 Partial derivatives

Given the definition of g_H and g_L, their partial derivatives are given by

$$\begin{aligned}
\frac{\partial g_H}{\partial X_H} &= (1-S)\left[2\zeta - (\varepsilon + \gamma X_H) + (1-X_H)\gamma\right] - 1 \\
\frac{\partial g_L}{\partial X_H} &= -q_L(BB) - \delta(1-S)(1-X_H)1_{\{q_L(BB)>0\}} \\
\frac{\partial g_H}{\partial X_L} &= -\left[S(2\beta X_H + \delta X_B) + (1-S)(\varepsilon + \gamma X_H)\right] \\
\frac{\partial g_L}{\partial X_L} &= 2q_L(LB) - q_L(BB) + \gamma S(1-X_H)1_{\{q_L(BB)>0\}} - 1 \\
\frac{\partial g_H}{\partial S} &= -2\zeta X_H - (\varepsilon + \gamma X_H)(1-X_H) \\
\frac{\partial g_L}{\partial S} &= (1-X_H)(\varepsilon + \delta X_H)1_{\{q_L(BB>0)\}}.
\end{aligned}$$

Note, if $X_H \geq X_H^{\Delta}$, then $q_L(CC) = 0$ and hence $\partial g_L/\partial X_H = \partial g_L/\partial S = 0$.

A.2 Proof of Lemma 2.3.1.2

Since $\eta = 0$, every constellation with $X_H + X_L = 1$, which implies $X_B = 0$, is a steady state ($\dot{X}_H = \dot{X}_L = 0$). In the following we investigate their stability. Let f_{ll} denote the matrix

$$f_{ll} = \begin{pmatrix} \frac{\partial \dot{X}_H}{\partial X_H} & \frac{\partial \dot{X}_H}{\partial X_L} \\ \\ \frac{\partial \dot{X}_L}{\partial X_H} & \frac{\partial \dot{X}_L}{\partial X_L} \end{pmatrix}$$

and define $a := X_H(1-2q_H(HB))$ and $b := X_L(1-2q_L(LB))$. For $X_B = 0$ the matrix f_{ll} equals $\left(\begin{smallmatrix} a & a \\ b & b \end{smallmatrix}\right)$ and has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a+b$. If $X_H = 1$ and hence $X_L = 0$ the non positivity of $a+b = a$ is equivalent to $S \leq 1 - 1/2\zeta$. If in contrast $X_H = 0$ and $X_L = 1$ we need for stability that $a + b = b \leq 0$. This is equivalent to $S \geq 1/2\zeta$ and can not be true since $S < 1/2$ and $\zeta \leq 1$. If $X_H, X_L > 0$ we have

$$a + b = 1 - 2(X_H q_H(HB) + X_L q_L(LB)).$$

For $x \in [0,1]$, consider the function

$$h(x) = x q_H(HB; x, 1-x, S) + (1-x) q_L(LB; x, 1-x, S).$$

Then stability, i.e. $a + b \leq 0$, is equivalent to $h(X_H) \geq 1/2$. We will investigate the four possible cases seperately. If $q_H(HB) = q_L(LB) = 0$, then $h = 0$. So we can except this first case. As a second case let $q_H(HB) = 0$ and $q_L(LB) > 0$. Then,

$$f(x) = (1-x)(\zeta S - \beta(1-S)x) \leq (1-x)\zeta S < 1/2, \tag{0.209}$$

since $S < 1/2$ and $(1-x), \zeta < 1$. Thus, we can exclude this case as well. As a third case let $q_H(HB) > 0$ and $q_L(LB) = 0$. Here,

$$f(x) = (1-x)(\zeta(1-S) - \beta S(1-x) = x\zeta - S(x\zeta + \beta x(1-x)).$$

To get $f(X_H) \geq 1/2$ we need $X_H \geq 1/2$. Then, $f(X_H) \geq 1/2$ yields

$$S \leq \frac{X_H \zeta - 1/2}{X_H \zeta + \beta X_H (1 - X_H)}.$$

The right hand side of the last inequality is increasing in X_H for $X_H \geq 1/2$. Hence, to achieve $f(X_H) \geq 1/2$ we need at least

$$S \leq \frac{\zeta - 1/2}{\zeta} = 1 - \frac{1}{2\zeta}.$$

In case 4 we have $q_H(HB), q_L(LB) > 0$. Here, f is a convex function in x:

$$f(x) = \zeta S + (\zeta - 2\zeta S - \beta)x + \beta x^2.$$

Hence, for all $0 < x < 1$, $f(x) \leq \max\{f(0), f(1)\}$. We have $f(0) = \zeta S < 1/2$ and $f(1) = \zeta(1-S)$. For $S > 1 - 1/2\zeta$, $f(1) < 1/2$. Summarizing we can see that in the first two cases no stable steady state exists, while in the last two cases a necessary condition for stability is given by $S \leq 1 - 1/2\zeta$. □

A.3 Proof of Lemma 2.4.1.1

For $S \in (\underline{S}, \min\{\overline{S}, \widetilde{S}\}]$ let $X_H = X_H^*(S)$, while $X_L = 0$.

Case 1: $\overline{S} \leq \widetilde{S}$
Set $S = \overline{S}$. The stationarity of λ_H yields

$$0 = \left(r - \theta X_B \frac{\partial g_H}{\partial X_H} \right) \lambda_H + k - \frac{\xi}{(1 - X_H)^{1-\xi}}.$$

Note that for $X_H = X_H^*(S)$ it is easy to check that $\partial g_H / \partial X_H \leq 0$. To achieve stationarity of λ_S, we have to find a $\lambda_S \geq 2 \frac{2\nu+\mu}{\nu\mu}(1 - X_H)^\xi$ such that

$$0 = \dot{\lambda}_S = -\theta X_B \lambda_H \frac{\partial g_H}{\partial S} + \lambda_S(r + \mu + 2\nu).$$

Since $\lambda_H < 0$ increases in k and $\frac{\partial g_H}{\partial S} < 0$, the solution to the above linear equation is sufficiently large, if k is sufficiently large.

Case 2: $\widetilde{S} < \overline{S}$
Here the stationarity of λ_H yields

$$0 = \left(r - \theta X_B \frac{\partial g_H}{\partial X_H} \right) \lambda_H + k - \frac{\xi}{(1 - X_H)^{1-\xi}} s^*(S),$$

and $\lambda_S = \lambda_S(S)$ is given by (2.20). We seek for a proper S such that $\dot{\lambda}_S = 0$ holds, cf. (2.22), where $\partial g_L / \partial S = 0$. If the first summand of (2.22) is denoted by $f_1(S)$ and the second one by $f_2(S)$, then we aim to solve $-f_1(S) = f_2(S)$. It is easy to check that at $\underline{S}$ (note, $X_H^*(\underline{S}) = 1$) we have $f_1(\underline{S}) = 0$. Depending on ξ it holds $f_2(\underline{S}) > 0$ (for $\xi = 0$) or $f_2(\underline{S}) = 0$ (for $\xi > 0$). Furthermore, $f_2(S) \to \infty$ for $S \to 1/2$, while $-f_1$ is bounded. Since f_2 is independent of the parameter k while $-f_1$ is growing linearly in k, we get for sufficiently large k that $-f_1(S) > f_2(S)$ for some relevant S. Summarizing we have for sufficiently large k: $-f_1(\underline{S}) \leq f_2(\underline{S})$, $-f_1(S) > f_2(S)$ for some $S \in (\underline{S}, 1/2)$, $f_2(1/2) = \infty$, $-f_1(1/2) < \infty$ and f_1, f_2 are continuous functions on $(\underline{S}, 1/2)$. Hence, there exists at least one intersection between the two functions in the interval $(\underline{S}, 1/2)$. □

B Appendix Section 3

Families gain utility from transmitting cultural identity as well as transmitting language(s) with high practical communication advantage (ϕ) and status advantage (ω). We number families by an index i, and denote the family type of family i by F_i. Let $I_F := \{i \mid F_i = F\}$ be the set of all families of type F. Based on the approach taken in Wickström (2005), we assume the utility of family i from transmitting repertoire R to be of the form $u_i(F; R) = u_i(F; R, \phi(R), \omega(R))$ with $\partial u / \partial \phi > 0$ and $\partial u / \partial \omega > 0$. Preferences are randomly distributed and the preference structure is assumed to be stable and stationary over time and generations.

We further specify

$$\omega(R) := \begin{cases} g(2S) & : \quad R = L \\ g(1) & : \quad R = H, B \end{cases} \tag{0.210}$$

$$\phi(R) := \begin{cases} h(1 - X_H) & : \quad R = L \\ h(1 - X_L) & : \quad R = H \\ h(1) & : \quad R = B \end{cases} \tag{0.211}$$

g and h being some monotonously increasing functions. Note, we use $2S$ since $S \leq 1/2$. This captures the communication and the status advantage. To obtain assumptions A2 and A3, we set $u_i(HH, L) = u_i(HB, L) = 0$, $u_i(LL, H) = u_i(LB, H) = 0$, $u_i(HH, B) = 0$ and $u_i(LL, B) = 0$.

Parents choose the language repertoire R that maximizes their utility.

To get the distribution of language repertoires among all children (the next generation), we need to aggregate over i. For repertoires $R_1, R_2, R_3 \in H, L, B$ with $R_1 \neq R_2 \neq R_3$ consider

$$\xi_{R_1}(F) := \#\left\{i \in I_F \,|\, u_i(F; R_1) > u_i(F; R) \; \forall R \neq R_1\right\} \tag{0.212}$$

$$\xi_{R_1,R_2}(F) := \#\left\{ i \in I_F \;\middle|\; \begin{array}{l} u_i(F; R_1) = u_i(F; R_2) \wedge \\ u_i(F; R_1) > u_i(F; R) \; \forall R \neq R_1, R_2 \end{array} \right\} \tag{0.213}$$

$$\xi_{R_1,R_2,R_3}(F) := \#\left\{i \in I_F \,|\, u_i(F; R_1) = u_i(F; R_2) = u_i(F; R_3)\right\}, \tag{0.214}$$

where $\#$ denotes the relative measure. By construction, $1 = \sum_G \xi_G(F)$, for $G \in \{H, L, B, HL, HB, LB, HLB\}$. We define $q_{LR}(F)$ via

$$q_H(F) := \xi_H(F) + \frac{1}{2}\big(\xi_{H,L}(F) + \xi_{H,B}(F)\big) + \frac{1}{3}\xi_{H,L,B}(F), \tag{0.215}$$

and $q_L(F), q_B(F)$ analogously. Note, in formulas (0.212)-(0.215) we suppressed the dependencies on ϕ and ω, or X and $S(L)$ respectively. From the construction of $q_R(F)$ and (0.210)-(0.211) one can deduce properties P1-P2 stated above (simply by counting).

C Appendix Section 4

Consider a population of size $\mathcal{N}$, where $\mathcal{N}$ is large. We assume that half of the population is female and that the distribution of language repertoires is identical for both sexes. Then, $N_R/2$ is the number of female individuals with language repertoire R. The distribution of repertoires is given by $X_H = N_H/\mathcal{N}$, $X_L = N_L/\mathcal{N}$ and $X_B = N_B/\mathcal{N}$.

Family formation is conceptualized as a repeated random procedure of choosing pairs. Let us start with one such pair consisting of a female Y and a male Z. The probabilities that a certain pair is chosen depend on the numbers of speakers as well as on linguistic concentration. For maximal concentration, i.e. $C = 1$, H monolinguals only meet other H monolinguals and speakers of L only meet other speakers of L. For $C = 0$ meeting probabilities only depend on the sizes of the language groups. In between these two boundary cases we simply use linear interpolation. First, we derive the relevant conditional probabilities. In terms of conditional probability $C = 0$ translates to stochastic independence of Y and Z. Let $R, R' \in \{L, B\}$. We obtain

$$\mathbb{P}[Z = H|Y = H] = C + (1 - C)\mathbb{P}[Z = H] = C + (1 - C)X_H \tag{0.216}$$

$$\mathbb{P}[Z = R|Y = H] = (1 - C)\mathbb{P}[Z = R] \qquad = (1 - C)X_R \tag{0.217}$$

If both speak the minority language, things are slightly more complicated. For $C = 0$ we have $\mathbb{P}[Z = R|Y = R'] = \mathbb{P}[Z = R] = X_R$. For $C = 1$ we get $\mathbb{P}[Z = R|Y = R'] = \mathbb{P}[Z = R]/\mathbb{P}[Z = L, B] = X_R/(1 - X_H)$. Linear interpolation yields

$$\mathbb{P}[Z = R|Y = R'] = X_R \left(1 + C\frac{X_H}{1 - X_H}\right) \tag{0.218}$$

Next, we use that $\mathbb{P}[Y = R_1, Z = R_2] = \mathbb{P}[Y = R_1]\mathbb{P}[Z = R_2|Y = R_1]$ and that $\mathbb{P}[Y = R_1] = X_{R_1}$, R_1 and R_2 being any of the three language repertoires H, L or B. Note that a family type R_1R_2, $R_1 \neq R_2$, is obtained either by $Y = R_1$, $Z = R_2$ or by $Y = R_2$; $Z = R_1$ (for the family type we do not take the sex of the parents into account). Hence, the probability to obtain a pair of type R_1R_2, denoted by $\mathbb{P}[R_1R_2]$ is given by $\mathbb{P}[Y = R_1, Z = R_2]$, if $R_1 = L_2'$, and by $2\mathbb{P}[Y = R_1, Z = R_2]$, if $R_1 \neq R_2$.

All $N/2$ pairs are chosen randomly after one another. The total expected number of RR'-type pairs equals $N/2 \cdot \mathbb{P}[R_1R_2]$ and hence the expected fraction of R_1R_2-type pairs is $\mathbb{P}[R_1R_2]$. We assume that all couples of types other than HL form families. As was previously mentioned, we assume that parents shall be able to properly communicate with each other, and therefore we exclude HL families. Hence, after this first step we have $N/2$ pairs with $N(1-C)X_HX_L$ of them being of type HL. Splitting these HL pairs again and repeating the random selection we obtain new pairs of types HH, LL and HL. This procedure is repeated until only HH and LL pairs remain. This way, half of the $N(1 - C)X_HX_L$ pairs of type HL will be transformed into HH pairs, while the other half will form LL pairs. As a result, we obtain the numbers presented in equations (4.42)-(4.46). Note, due to the law of large numbers ($\mathcal{N}$ is assumed to be large), the realized number of R_1R_2-type of pairs can be approximated by the expected number.

D Appendix Section 5

D.1 1G-1L model

Proof of thoerem 5.2.2.1

We prove (5.68)-(5.70) by a formalization of the repeated (multiple rounds) 2-step couple/family formation process. Prior to the first round, the distribution of repertoires among all individuals who are not part of a couple is given by $Y_0 := (Y_{0,H}, Y_{0,L}) = (X_H, X_L)$. Next, all individuals become part of a couple. The distribution of couple types after step 1 of round 1 is given by

$$\psi_{1,HH} = Y_{0,H}^2 + CY_{0,H}Y_{0,L} \tag{0.219}$$
$$\psi_{1,HL} = 2(1-C)Y_{0,H}Y_{0,L} \tag{0.220}$$
$$\psi_{1,LL} = Y_{0,L}^2 + CY_{0,H}Y_{0,L}. \tag{0.221}$$

By assumption all HH and LL couples are successful but a fraction p of all HL couples. The remaining $1-p$ HL couples enter a second round of couple and family formation. With respect to the total population, $(1-p)\psi_{0,HL}$ of all couples split up again. After round one we thus have the following family distribution

$$\Psi_{1,HH} = X_H^2 + CX_HX_L \tag{0.222}$$
$$\Psi_{1,HL} = 2p(1-C)X_HX_L \tag{0.223}$$
$$\Psi_{1,LL} = X_H^2 + CX_HX_L \tag{0.224}$$
$$\Psi_{1,non} = 2(1-p)(1-C)X_HX_L \tag{0.225}$$

Half of these individuals who are not part of a family ($\Psi_{1,non}$) are H monolinguals, the other half are L monolinguals. In round 2, they again form couples, so that

$$\psi_{2,HH} = (1+C)/4 \tag{0.226}$$
$$\psi_{2,HL} = (1-C)/2 \tag{0.227}$$
$$\psi_{2,LL} = (1+C)/4. \tag{0.228}$$

Again, p of all these HL couples form families, the others split up. This process is repeated over and over again. After round $i+1$ we have the distribution

$$\Psi_{i+1,HH} = \Psi_{i,HH} + \frac{1+C}{4}\Psi_{i,non} \tag{0.229}$$
$$= \Psi_{1,HH} + \frac{1+C}{4}\Psi_{1,non}\sum_{j=0}^{i-1}\left(\frac{(1-p)(1-C)}{2}\right)^j \tag{0.230}$$
$$\Psi_{i+1,HL} = \Psi_{i,HL} + p\frac{1-C}{2}\Psi_{i,non} \tag{0.231}$$
$$= \Psi_{1,HL} + p\frac{1-C}{2}\Psi_{1,non}\sum_{j=0}^{i-1}\left(\frac{(1-p)(1-C)}{2}\right)^j \tag{0.232}$$
$$\Psi_{i+1,LL} = \Psi_{i,LL} + \frac{1+C}{4}\Psi_{i,non} \tag{0.233}$$
$$= \Psi_{1,LL} + \frac{1+C}{4}\Psi_{1,non}\sum_{j=0}^{i-1}\left(\frac{(1-p)(1-C)}{2}\right)^j \tag{0.234}$$
$$\Psi_{i+1,non} = (1-p)\frac{1-C}{2}\Psi_{i,non} \tag{0.235}$$
$$= \left(\frac{(1-p)(1-C)}{2}\right)^i\Psi_{1,non} \tag{0.236}$$

Considering the limit $i \to \infty$ and plugging in (0.222)-(0.225), we obtain the results stated above. $\square$

D.2 3G-1L model

Proof of Lemma 5.3.1.1

To see that these steady states are stable, we express the differential equation (5.118)-(5.120) by $\dot{\eta}(t) = F\eta(t) + u$, with column vectors $\eta(t) = (n(t), \mathcal{N}(t), \mathfrak{N}(t))'$ and $u = (m, \mathcal{M}, \mathfrak{M})'$, and matrix

$$F = \begin{pmatrix} -1/20 & \lambda_{\mathcal{X}} & 0 \\ 1/20 & -1/30 & 0 \\ 0 & \lambda_{\mathcal{X}} & -\mu_{\mathfrak{X}} \end{pmatrix}.$$

The Eigenvalues of F are $-\mu_{\mathfrak{X}}$ and $(-5 \pm \sqrt{1 + 720\lambda_{\mathcal{X}}})/120$. For $\lambda_{\mathcal{X}} < 1/30$ all three eigenvalues are negative, and hence the system is asymptotically stable. Solving $F\eta + u = 0$ we obtain the steady states from the lemma. $\square$

Proof of Lemma 5.3.1.2

For the sake of the this proof we write λ for $\lambda_{\mathcal{X}}$ and μ for $\mu_{\mathfrak{X}}$. We have

$$(d/dt)(n(t) + \mathcal{N}(t)) = (\lambda - 1/30)\mathcal{N}(t) + \mathcal{M}.$$

Hence, for $\lambda \geq 1/30$, $n(t) + \mathcal{N}(t)$ goes to infinity as $t \to \infty$. From this is can be seen from the structure of the system that $n(t), \mathcal{N}(t), \mathfrak{N}(t)$ all tend to infinity.

In the proof of Lemma 5.3.1.1 we used that $\eta(t) = (n(t), \mathcal{N}(t), \mathfrak{N}(t))'$ evolves according to $\dot{\eta}(t) = F\eta(t) + u$, where F is a matrix with real Eigenvalues $r_1 = -\mu < 0$, $r_2 = (-5 - \sqrt{1 + 720\lambda})/120 < 0$ and $r_3 = (-5 + \sqrt{1 + 720\lambda})/120$. For $\lambda > 1/30$ the Eigenvalue r_3 is positive and for $\lambda = 1/30$ we have $r_3 = 0$. For $\lambda < 1/30$ and $\mu \neq r_2$ we have three distinct eigenvalues and the solution η is of the form

$$\eta(t) = C_1 v_1 e^{r_1 t} + C_2 v_2 e^{r_2 t} + C_3 v_3 e^{r_3 t} + c_4,$$

where $C_1, ..., C_3 \in \mathbb{R}$, $c_4 \in \mathbb{R}^2$ and Eigenvectors $v_1, ..., v_3$. For $\mu = r_2$ the solution has a comparable form and for $\lambda = 1/30$ there are additional linear terms $at + b$. Since $r_1, r_2 < 0$, we can conclude from the general form of $\eta(t)$, that $n(t)/\mathcal{N}(t)$, $\mathcal{N}(t)/\mathfrak{N}(t)$, $\dot{n}(t)/\dot{\mathcal{N}}(t)$ and $\dot{\mathcal{N}}(t)/\dot{\mathfrak{N}}(t)$ converge as $t \to \infty$. Hence, we can use l'Hospital and calculate the limits as follows:

$$\begin{aligned}
\lim_{t\to\infty}\frac{\mathcal{N}(t)}{n(t)} &= \lim_{t\to\infty}\frac{\dot{\mathcal{N}}(t)}{\dot{n}(t)} \\
&= \lim_{t\to\infty}\frac{-\mathcal{N}(t)/30+n(t)/20+M}{\lambda_{\mathcal{X}}\mathcal{N}(t)-n(t)/20+m} \\
&= \lim_{t\to\infty}\frac{-\mathcal{N}(t)/(30n(t))+1/20}{\lambda_{\mathcal{X}}\mathcal{N}(t)/n(t)-1/20} \\
&= \frac{\frac{1}{20}-\frac{1}{30}\lim_{t\to\infty}\frac{\mathcal{N}(t)}{n(t)}}{-\frac{1}{20}+\lambda_{\mathcal{X}}\lim_{t\to\infty}\frac{\mathcal{N}(t)}{n(t)}}
\end{aligned}$$

The only non-negative solution to this equation is $L_{\mathcal{N}/n}$. Moreover,

$$\begin{aligned}
\lim_{t\to\infty}\frac{\mathcal{N}(t)}{\mathfrak{N}(t)} &= \lim_{t\to\infty}\frac{\dot{\mathcal{N}}(t)}{\dot{\mathfrak{N}}(t)} \\
&= \lim_{t\to\infty}\frac{-\mathcal{N}(t)/30+n(t)/20+M}{\mathcal{N}(t)/30-\mu_{\mathfrak{X}}\mathfrak{N}(t)\mathfrak{M}} \\
&= \lim_{t\to\infty}\frac{-1/30+(1/20)n(t)/\mathcal{N}(t)}{1/30-\mu_{\mathfrak{X}}\mathfrak{N}(t)/\mathcal{N}(t)}
\end{aligned}$$

and hence

$$-\mu_{\mathfrak{X}}+L_{\mathcal{N},\mathfrak{N}}/30=-1/30+L_{n,\mathcal{N}}/20.$$

□

D.3 10G-1L model

Proof of Lemma 5.3.2.1

The system is of the form $\dot{v}(t) = F \cdot v(t) + u$, where v is the column verctor $(N_1, ..., N_{10})'$. The eigenvalues of the matrix F are $-(1 + 9\mu)/10$ and all five solutions z of the polynomial equation $h(z) = 0$ with

$$h = 1-30\lambda+(50-300\lambda)z+(1000-1000\lambda)z^2+10000z^3+50000z^4+100000z^5.$$

Using the corresponding Routh-Hurwitz table, see e.g. Golnaraghi & Kuo (2017), we show that all the roots of h have a negative real part.

10^5	10^4	$50 - 300\lambda$
$10^5/2$	$(1-\lambda)10^3$	$1 - 30\lambda$
b_1	b_2	0
c_1	c_2	0
d_1	0	0
e_1	0	0

with

$$
\begin{aligned}
b_1 &= 10^4 - 2(1-\lambda)10^3 & b_2 &= 48 - 240\lambda \\
c_1 &= \left[(1-\lambda)10^7 - 2(1-\lambda)^2 10^6 - (24 - 120\lambda)10^5\right]/b_1 & c_2 &= 1 - 30\lambda \\
d_1 &= (c_1 b_2 - b_1 c_2)/c_1 \\
e_1 &= 1 - 30\lambda
\end{aligned}
$$

Since $\lambda > 0$, we have $b_1 > 0$. Due to $\lambda < 1/30$, we get $b_1 \cdot c_1 > 29/30 \cdot 10^7 - 2 \cdot 10^6 - 24 \cdot 10^5 > 0$. Going even futher we have $c_1 > 4 \cdot 10^2$. Moreover, $b_2 > 40$, $b_1 < 10^4$ and $c_2 < 1$. Hence, $c_1 b_2 - b_1 c_2 > 1.6 \cdot 10^4 - 10^4 > 0$. Since all entries in the first column have a positive sign, all the (real and complex) eigenvalues of F have negative real part. As a result, the system is stable.

We continue with deriving the actual steady states. For $i = 2, ..., 10$, the above differential equations are of the form

$$\dot{N}_i = -A_i N_i + B_i N_{i-1} + M_i,$$

where A_i and B_i are positive constants. For $i = 2, ..., 5$ we have $A_i = B_i = 1/10$, for $i = 6$ we have $A_i = \mu + (1-\mu)/10$ and for $i = 7, ..., 10$ we have $A_i = \mu + (1-\mu)/10$ and $(1-\mu)/10$. Solving $\dot{N}_i = 0$ for $i = 2, ..., 10$, we get

$$N_{i,\infty} = \left(\prod_{j=2}^{i} \frac{B_j}{A_j}\right) N_{1,\infty} + \sum_{k=2}^{i-1} \left(\prod_{j=k+1}^{i} \frac{B_j}{A_j}\right) \frac{M_k}{A_k} + \frac{M_i}{A_i}. \tag{0.237}$$

To derive $N_{1,\infty}$, we first express $\mathcal{N}_\infty := \sum_{i=3}^{5} N_{i_\infty}$ in terms of $N_{1,\infty}$. We have, $q_i = M_i/M$,

$$
\begin{aligned}
\mathcal{N}_\infty &= \left(\sum_{i=3}^{5} \prod_{j=2}^{i} \frac{B_j}{A_j}\right) N_{1,\infty} + \sum_{i=3}^{5} \left[\sum_{k=2}^{i-1} \left(\prod_{j=k+1}^{i} \frac{B_j}{A_j}\right) \frac{q_k}{A_k} + \frac{q_i}{A_i}\right] M \\
&= \mathcal{D} N_{1,\infty} + \mathcal{E} \cdot M.
\end{aligned} \tag{0.238}
$$

Due to $\cdot N_1 = -N_1/10 + \lambda\mathcal{N} + M_1$ we obtain

$$N_{1,\infty} = \frac{10}{1-10\lambda\mathcal{D}}\big(\lambda\mathcal{E}M + M_1\big). \tag{0.239}$$

Using D_i and E_i as defined in the lemma, we obtain the proposed steady states.

□

Proof of Lemma 5.3.2.2

This proof is analogous to the proof of the analogous lemma for the 3G model. Hence, we only provide a short version of the proof. Let

$$W_{i,i+1} := \lim_{t\to\infty} \frac{N_i(t)}{N_{i+1}(t)}.$$

Using the l'Hospital, we get for $i = 2, 3, 4$ that

$$W_{i,i+1} = \lim_{t\to\infty} \frac{-N_i(t) + N_{i-1}(t) + 10M_i}{-N_{i+1}(t) + N_i(t) + 10M_{i+1}} = \frac{-1 + W_{i-1,i}}{-W_{i,i+1}^{-1} + 1}$$

Hence, $W_{i,i+1} = W_{i-1,i}$. Next, we consider $W_{1,2}$:

$$\begin{aligned} W_{1,2} &= \lim_{t\to\infty} \frac{-N_1(t) + 10\lambda\mathcal{N} + 10M_1}{-N_2(t) + N_1(t) + 10M_2} \\ &= \lim_{t\to\infty} \frac{-N_1(t)/N_2(t) + 10\lambda\big(N_3(t)/N_2(t) + N_4(t)/N_2(t) + N_5(t)/N_2(t)\big)}{-1 + N_1(t)/N_2(t)} \\ &= \frac{-W_{1,2} + 10\lambda\big(W_{1,2}^{-1} + W_{1,2}^{-2} + W_{1,2}^{-3}\big)}{-1 + W_{1,2}}. \end{aligned}$$

Hence, $W_{1,2}$ is a solution of the polynomial equation $0 = s(x) = x^5 - 10\lambda(x^2 + x + 1)$. We have $s(0) = -10\lambda < 0$, $s'(x) = 5x^4 - 10\lambda(2x+1)$, $s'(0) = -10\lambda < 0$ and $s''(x) = 20(x^3 - \lambda)$. It is easy to see now that the function s has a unique root on the positive real line $\mathbb{R}_+$. This unique root is called W_∞. Hence, $W_{i,i+1} = W_\infty$ for $i = 1, ..., 4$. Next, we consider $W_{5,6}$ and $W_{6,7}$:

$$W_{5,6} = \lim_{t\to\infty} \frac{-N_5(t) + N_4(t) + 10M_5}{-(1+9\mu)N_6(t) + N_5(t) + 10M_6} = \frac{-1 + W_\infty}{-(1+9\mu)W_{5,6}^{-1} + 1}.$$

Therefore, $W_{5,6} = 9\mu + L_\infty$. Moreover,

$$\begin{aligned} W_{6,7} &= \lim_{t\to\infty} \frac{-(1+9\mu)N_6(t) + N_5(t) + 10M_6}{-(1+9\mu)N_7(t) + (1-\mu)N_6(t) + 10M_7} \\ &= \frac{-(1+9\mu) + W_{5,6}}{-(1+9\mu)W_{6,7}^{-1} + (1-\mu)}. \end{aligned}$$

Hence, $W_{6,7} = W_{5,6}/(1-\mu) = (W_\infty + 9\mu)/(1-\mu)$. Last we consider $W_{j,j+1}$ for $j = 7, 8, 9$:

$$\begin{aligned} W_{j,j+1} &= \lim_{t\to\infty} \frac{-(1+9\mu)N_j(t) + (1-\mu)N_{j-1}(t) + 10M_j}{-(1+9\mu)N_{j+1}(t) + (1-\mu)N_j(t) + 10M_{j+1}} \\ &= \frac{-(1+9\mu) + (1-\mu)W_{j-1,j}}{-(1+9\mu)W_{j,j+1}^{-1} + (1-\mu)}. \end{aligned}$$

Hence, for $j = 7, 8, 9$ we have $W_{j,j+1} = W_{j-1,j} = W_{6,7}$. □

D.4 1G-nL model

Proof of Lemma 5.4.1.1 (Couple distribution ψ_F)

Recall, $X_i = X_{L_i} + X_{B_i}$. Let $R \in \mathbb{F}$, $R \neq L_i, B_i$. As for the 1G-1L model, cf. (5.63) and (5.64), we use linear interpolation and get

$$\mathbb{P}[L_i \vee B_i \,|\, L_i] = \mathbb{P}[L_i \vee B_i \,|\, B_i] \;=\; C_{1,i} + (1 - C_{1,i})X_i \tag{0.240}$$

$$\mathbb{P}[R \,|\, L_i] = \mathbb{P}[R \,|\, B_i] \;=\; (1 - C_{1,i})X_R \tag{0.241}$$

Next, we use that

$$\mathbb{P}[R] = \mathbb{P}[R|H]\mathbb{P}[H] + \sum_{j=1}^{n} \mathbb{P}[R|L_j]\mathbb{P}[L_j] + \mathbb{P}[R|B_j]\mathbb{P}[B_j].$$

Therefore,

$$\begin{aligned}\mathbb{P}[L_i \vee B_i \,|\, H] \cdot X_H &= X_i - \big(C_{1,i} + (1 - C_{1,i})X_i\big)X_i - \sum_{j=1, j\neq i}^{n} (1 - C_{1,j})X_iX_j \\ &= X_i \left((1 - C_{1,i}) - \sum_{j=1}^{n} (1 - C_{1,j})X_j \right) \end{aligned} \tag{0.242}$$

$$\begin{aligned}\mathbb{P}[H \,|\, H] &= 1 - \sum_{j=1}^{n} (1 - C_{1,j})X_j \\ &= X_H + \sum_{j=1}^{n} C_jX_j. \end{aligned} \tag{0.243}$$

More precisely,

$$\mathbb{P}[L_i \,|\, H] \cdot X_H = X_{L_i} \left((1 - C_{1,i}) - \sum_{j=1}^{n} (1 - C_{1,j})X_j \right) \tag{0.244}$$

$$\mathbb{P}[B_i \,|\, H] \cdot X_H = X_{B_i} \left((1 - C_{1,i}) - \sum_{j=1}^{n} (1 - C_{1,j})X_j \right) \tag{0.245}$$

Finally, we use $\mathbb{P}[R_1R_1] = \mathbb{P}[R] \mp [R_1|R_1]$ and $\mathbb{P}[R_1R_2] = \mathbb{P}[R_1]\mathbb{P}[R_2|R_1] + \mathbb{P}[R_2]\mathbb{P}[R_1|R_2]$ to obtain the couple distribution stated in (5.172)-(5.178). To determine $\psi_{L_iL_i}$, $\psi_{L_iB_i}$ and $\psi_{B_iB_i}$, we have to consider Y_{L_i}, resp. Y_{B_i}, the fraction of the population monolingual in L_i, resp. bilingual in L_i and H, and not in a couple with a non-L_i speaker. Recall, $E_i = C_{1,i} + (1 - C_{1,i})X_i$. We have

$$\begin{aligned} Y_{L_i} &= X_{L_i} - \frac{1}{2}\psi_{H,L_i} - \frac{1}{2}\sum_{j\neq i} \psi_{L_i,L_j} - \frac{1}{2}\sum_{j\neq i} \psi_{L_i,B_j} \\ &= X_{L_i} \left(1 - \frac{1}{2}(1 - C_{1,i})(1 + X_H) \right) \\ &\quad + X_{L_i} \left(\frac{1}{2}\sum_{j=1}^{n} (1 - C_{1,j})X_j - \frac{1}{2}\sum_{j\neq i} (2 - C_{1,i} - C_{1,j})X_j \right) \\ &= E_i \cdot X_{L_i}. \end{aligned}$$

Similarly, $Y_{B_i} = E_i \cdot X_{B_i}$. We set $Y_i = Y_{L_i} + Y_{B_i} = E_i \cdot X_i$. As in the 1L model, we have

$$\begin{aligned}
\psi_{L_i,L_i} &= \frac{1}{Y_i}\big(Y_{L_i}^2 + C_{2,i} Y_{L_i} Y_{B_i}\big) \\
\psi_{L_i,L_i} &= \frac{2}{Y_i}\big(1 - C_{2,i}\big) Y_{L_i} Y_{B_i} \\
\psi_{B_i,B_i} &= \frac{1}{Y_i}\big(Y_{B_i}^2 + C_{2,i} Y_{L_i} Y_{B_i}\big)
\end{aligned}$$

Plugging in $Y_{L_i} = E_i X_{L_i}$ and $Y_{B_i} = E_i X_{B_i}$, we obtain the distribution stated in the lemma. □

Derivation of properties of the transmission functions $q_R(F)$ depicted in Table 10

Here we extend the utility maximization approach suggested in Wickström (2005) and adapted in Templin (2018) to the case of multiple minorities. Accordingly, this proof can be seen as an extension of the related proof for the 1G-1L case, cf. Appendix B. Families gain utility from transmitting cultural identity as well as transmitting language(s) with high practical communication advantage (ϕ) and status advantage (ω). We number families by an index k, and denote the family type of family k by F_k. Let $I_F := \{k \mid F_k = F\}$ be the set of all families of type F. We assume the utility of family k from transmitting repertoire R to be of the form $u_k(F;R) = u_k(F;R,\phi(R),\omega(R))$ with $\partial u/\partial\phi \geq 0$ and $\partial u/\partial\omega \geq 0$. Preferences are randomly distributed and the preference structure is assumed to be stable and stationary over time and generations. We define $X_L = \sum_i X_{L_i}$ and specify

$$\omega(R) := \begin{cases} g(2S_i) & : \quad R = L_i \\ g(1) & : \quad R = H, B \end{cases} \tag{0.246}$$

$$\phi(R) := \begin{cases} h(X_{L_i} + X_{B_i}) & : \quad R = L_i \\ h(1 - X_L + X_{L_i}) & : \quad R = B_i \\ h(1 - X_L) & : \quad R = H \end{cases} \tag{0.247}$$

g and h being some monotonously increasing functions with

$$\partial\phi(x)/\partial x, \partial\omega(x)/\partial x \geq 0.$$

Note, we use $2S_i$ since $S_i \leq 1/2$. This captures the communication and the status advantage. To obtain assumptions A2 and A3, we set

$$
\begin{aligned}
0 &= u_k(HH, L_i) = u_k(HB_i, L_i) = u_k(HB_j, L_i) \\
&= u_k(L_jB_j, L_i) = u_k(B_jB_j, L_i), \\
0 &= u_k(L_iL_i, H) = u_k(L_iB_i, H), \\
0 &= u_k(HH, B_i) = u_k(L_iL_i, B_i) = u_k(L_jL_j, B_i) \\
&= u_k(L_jB_j, B_i) = u_k(B_jB_j, B_i).
\end{aligned}
$$

Parents choose the language repertoire R that yields the highest utility. We count the number of families for which the utility from one specific repertoire is higher than the utility from all other repertoires to obtain distribution of language repertoires among all children (aggregating over k). Let $\#$ denote the relative measure and consider

$$
\begin{aligned}
\xi_H(HB_i) &= \#\left\{k \in I_{HB_i} | u_k(HB_i; H) > u(HB_i; B_i)\right\} \\
\xi_{B_i}(HB_i) &= \#\left\{k \in I_{HB_i} | u_k(HB_i; H) < u(HB_i; B_i)\right\} \\
\xi_{HB_i}(HB_i) &= \#\left\{k \in I_{HB_i} | u_k(HB_i; H) = u(HB_i; B_i)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\xi_{L_i}(L_iB_i) &= \#\left\{k \in I_{L_iB_i} | u_k(L_iB_i; L_i) > u(L_iB_i; B_i)\right\} \\
\xi_{B_i}(L_iB_i) &= \#\left\{k \in I_{L_iB_i} | u_k(L_iB_i; L_i) < u(L_iB_i; B_i)\right\} \\
\xi_{L_iB_i}(L_iB_i) &= \#\left\{k \in I_{L_iB_i} | u_k(L_iB_i; L_i) = u(L_iB_i; B_i)\right\}
\end{aligned}
$$

$$\xi_H(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;H) > u(B_iB_i;L_i) \wedge \\ u(B_iB_i;H) > u(B_iB_i;B_i) \end{array} \right\}$$

$$\xi_{L_i}(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;L_i) > u(B_iB_i;H) \wedge \\ u(B_iB_i;L_i) > u(B_iB_i;B_i) \end{array} \right\}$$

$$\xi_{B_i}(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;B_i) > u(B_iB_i;H) \wedge \\ u(B_iB_i;B_i) > u(B_iB_i;L_i) \end{array} \right\}$$

$$\xi_{HL_i}(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;H) = u(B_iB_i;L_i) \wedge \\ u(B_iB_i;H) > u(B_iB_i;B_i) \end{array} \right\}$$

$$\xi_{HB_i}(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;H) > u(B_iB_i;L_i) \wedge \\ u(B_iB_i;H) = u(B_iB_i;B_i) \end{array} \right\}$$

$$\xi_{L_iB_i}(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;L_i) > u(B_iB_i;H) \wedge \\ u(B_iB_i;L_i) = u(B_iB_i;B_i) \end{array} \right\}$$

$$\xi_{HL_iB_i}(B_iB_i) = \#\left\{ k \in I_{B_iB_i} \;\middle|\; \begin{array}{l} u(B_iB_i;H) = u(B_iB_i;L_i) \wedge \\ u(B_iB_i;H) = u(B_iB_i;B_i) \end{array} \right\}$$

$$\xi_H(B_iB_j) = \#\left\{ k \in I_{B_iB_j} \;\middle|\; \begin{array}{l} u(B_iB_j;H) > u(B_iB_j;B_i) \wedge \\ u(B_iB_j;H) > u(B_iB_j;B_j) \end{array} \right\}$$

$$\xi_{B_i}(B_iB_j) = \#\left\{ k \in I_{B_iB_j} \;\middle|\; \begin{array}{l} u(B_iB_j;B_i) > u(B_iB_j;H) \wedge \\ u(B_iB_j;B_i) > u(B_iB_j;B_j) \end{array} \right\}$$

$$\xi_{HB_i}(B_iB_j) = \#\left\{ k \in I_{B_iB_j} \;\middle|\; \begin{array}{l} u(B_iB_j;B_j) > u(B_iB_j;H) \wedge \\ u(B_iB_j;B_j) > u(B_iB_j;B_i) \end{array} \right\}$$

$$\xi_{B_iB_j}(B_iB_j) = \#\left\{ k \in I_{B_iB_j} \;\middle|\; \begin{array}{l} u(B_iB_j;B_i) > u(B_iB_j;H) \wedge \\ u(B_iB_j;B_i) > u(B_iB_j;B_j) \end{array} \right\}$$

$$\xi_{HB_iB_j}(B_iB_j) = \#\left\{ k \in I_{B_iB_j} \;\middle|\; \begin{array}{l} u(B_iB_j;H) = u(B_iB_j;B_i) \wedge \\ u(B_iB_j;H) = u(B_iB_j;B_j) \end{array} \right\}$$

The sum over each of the above blocks equals 1. Assuming that the probabilities of language choice of indifferent parents are equally distributed between the languages repertoires we define $q_R(F)$ via

$$\begin{aligned}
q_H(HH) &= 1 \\
q_H(HB_i) &= \xi_H(HB_i) + \frac{1}{2}\xi_{HB_i}(HB_i) \\
q_H(B_iB_i) &= \xi_H(B_iB_i) + \frac{1}{2}\left(\xi_{HL_i}(B_iB_i) + \xi_{HB_i}(B_iB_i)\right) + \frac{1}{3}\xi_{HL_iB_i}(B_iB_i) \\
q_H(B_iB_j) &= \xi_H(B_iB_j) + \frac{1}{2}\big(\xi_{HB_i}(B_iB_j) + \xi_{HB_j}(B_iB_j)\big) + \frac{1}{3}\xi_{HB_iB_j}(B_iB_j),
\end{aligned}$$

$$\begin{aligned}
q_{L_i}(L_iL_i) &= 1 \\
q_{L_i}(L_iB_i) &= \xi_{L_i}(L_iB_i) + \frac{1}{2}\xi_{L_iB_i}(L_iB_i) \\
q_{L_i}(B_iB_i) &= \xi_{L_i}(B_iB_i) + \frac{1}{2}\left(\xi_{HL_i}(B_iB_i) + \xi_{L_iB_i}(B_iB_i)\right) + \frac{1}{3}\xi_{HL_iB_i}(B_iB_i)
\end{aligned}$$

and

$$
\begin{aligned}
q_{B_i}(HB_i) &= 1 - \alpha_H(HB_i) \\
q_{B_i}(L_iB_i) &= 1 - \alpha_{L_i}(L_iB_i) \\
q_{B_i}(B_iB_i) &= 1 - \alpha_H(B_iB_i) - \alpha_{L_i}(B_iB_i) \\
&= \xi_{B_i}(B_iB_i) + \frac{1}{2}\left(\xi_{HB_i}(B_iB_i) + \xi_{L_iB_i}(B_iB_i)\right) + \frac{1}{3}\xi_{HL_iB_i}(B_iB_i) \\
q_{B_i}(B_iB_j) &= \xi_{B_i}(B_iB_j) + \frac{1}{2}\big(\xi_{HB_i}(B_iB_j) + \xi_{B_iB_j}(B_iB_j)\big) + \frac{1}{3}\xi_{HB_iB_j}(B_iB_j).
\end{aligned}
$$

Let the utility functions $u_k(F; R, \phi, \omega)$ be differentiable in ϕ and ω with $\frac{\partial u}{\partial \phi}, \frac{\partial u}{\partial \omega} > 0$. Then, the signum of the partial derivatives of $q_{LR}(F)$ with respect to X_H, X_{L_i}, X_{B_i} are given by the first 4 columns of Table 10. (If $\partial u/\partial\phi > 0$, $\partial u/\partial\omega > 0$ and $\phi', \omega' > 0$ then we have >0 instead of ≥ 0). If, additionally, the derivatives of u with respect to ϕ, ω as well as the derivatives of ω, ϕ are constant and positive (affine functions), then we can also derive the partial derivatives of $q_{LR}(F)$ with respect to X_{L_j}, cf. the last column of Table 10.

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Erklärungen

Erklärung Ersteinreichung

Hiermit erkläre ich, dass ich mich noch nie einem Doktorexamen unterzogen habe und dass die vorgelegte Dissertation noch bei keiner anderen Fakultät oder einem ihrer Mitglieder vorgelegt wurde.

Erklärung Hilfe anderer und genutzte Hilfsmittel

Außer der in der Dissertation angeführten Literatur habe ich keine weiteren Hilfsmittel zu deren Erstellen verwendet. Bis auf das Essay mit dem Titel *Optimal language policy for the preservation of a minority language* in Abschnitt 2 der Dissertation habe ich die Dissertation ohne fremde Hilfe verfasst.

Das Essay in Abschnitt 2 wurde von mir als Hauptautoren mit der Hilfe von drei Koautoren erstellt. Diese drei sind

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Die ursprüngliche Idee zu dem Essay stammt im wesentlichen von Bengt-Arne Wickström und Gustav Feichtinger. Beide haben ein erstes grobes Konzept erarbeitet, das von Prof. Wickström und mir weiterentwickelt wurde. Ich habe die Literatur recherchiert und den Literaturüberblick verfasst. Das im Essay vorgeschlagene Modell wurde von Prof. Wickström und mir entwickelt (vgl. Abschnitt 2.2), und im überwiegen von mir analysiert (Abschnitte 2.3 und 2.4). Die numerische Analyse wurde parallel von mir und Dr. Seidl durchgeführt. Wir haben zwei verschiedene numerische Verfahren genutzt, sind zu den selben Ergebnissen gekommen und konnten damit die numerischen Resultate verifizieren. Die Ergebnisse haben alle AutorInnen zusammen diskutiert, und ich habe das Manuskript erstellt. Die Anteile an dem Essay sind übersichtlich in der folgenden Tabelle übersichtlich dargestellt.

Tätigkeit	Beteiligte Personen
Konzeption	Templin, Wickström, Feichtinger
Literaturrecherche und -überblick	Templin
Theoretische Ausarbeitung	Templin (überwiegend), Seidl, Wickström
Numerische Programmierung	Templin, Seidl
Ergebnisdiskussion	Templin, Wickström, Seidl
Erstellung des Manuskriptes	Templin
Finale Überarbeitung	Templin, Wickström, Seidl, Feichinger

Ich bezeuge durch meine Unterschrift, dass meine Angaben über die bei der Abfassung meiner Dissertation benutzten Hilfsmittel, über die mir zuteil gewordene Hilfe sowie über frühere Begutachtungen meiner Dissertation in jeder Hinsicht der Wahrheit entsprechen.

Berlin den

Torsten Templin